

Instanton representation of Plebanski gravity. VII:
Initial value and Gauss' law constraints in
rectangular form

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Abstract

We provide an algorithm for solving the initial value constraints of GR using the rectangular parametrization of the CDJ matrix. This paper focuses on the Gauss' law constraint and the criteria for its integrability. The Gauss' law constraint takes as its input the chosen configuration, and for an output reduces the CDJ matrix to its physical degrees of freedom. This in turn provides an input into the Hamiltonian constraint for which we provide a formal solution by expansion about the Kodama state in the general case. We establish criteria for integrability of the Gauss' law constraint for given starting configurations and we provide several examples.

1 Introduction

The canonical formulation of general relativity in the metric representation produces a totally constrained system as a consequence of diffeomorphism invariance. The Hamiltonian consists of a linear combination of first class constraints $H_\mu = (H, H_i)$, respectively the Hamiltonian and diffeomorphism constraints. These constraints H_μ have thus far turned out to be intractable in the metric representation due to their nonpolynomial structure in the basic variables. A major development occurred in 1988 with the introduction of the Ashtekar variables (see e.g. [1],[2],[3]), which led to a simplification of the initial value constraints into polynomial form. The basic phase space variables in the Ashtekar formalism are a left-handed $SU(2)_-$ Ashtekar connection A_i^a and its canonically conjugate variable, a densitized triad $\tilde{\sigma}_a^i$. The Ashtekar variables effectively enlarge the metric phase space Ω , essentially by embedding it into the phase space of a $SO(3)$ Yang–Mills theory. A remnant of this embedding is the inclusion of the Gauss’ law constraint G_a in the list of initial value constraints $H_\mu \rightarrow (H_\mu, G_a)$.

A general solution to the initial value constraints problem of general relativity entails finding a projection from the unconstrained phase space to the constraint surface defining the physical degrees of freedom. It is currently an open problem which has been transformed by various authors into a form more amenable to physical interpretation [4],[5]. It was shown in [6] that the algebraic subset of the initial value constraints H_μ could be solved by a substitution known as the CDJ Ansatz.

$$\tilde{\sigma}_a^i = \Psi_{ae} B_e^i. \quad (1)$$

The CDJ matrix Ψ_{ae} takes its values in two copies of $SU(2)_-$, and the magnetic field B_e^i must be nondegenerate. The constraints can in general be solved whether for just pure gravity, or to include matter couplings as demonstrated by Thiemann [4], by use of the CDJ Ansatz¹

The main obstacle to the claim of a complete solution to the initial value problem, as noted and by the interpretation of the present author, is the disparity in the form of the Gauss’ law constraint G_a in relation to the diffeomorphism and Hamiltonian constraints $H_\mu = (H, H_i)$. Whilst H_μ are simple algebraic equations, G_a is a set of differential equations. Neither has the existence of solutions to Gauss’ law conjunction with the remaining

¹The index convention is that lowercase Latin symbols a, b, c, \dots from the beginning of the alphabet will denote internal left-handed $SU(2)_-$ indices, and from the middle of the alphabet i, j, k will denote spatial indices. We will often denote a position vector in three space at fixed time t by \vec{r} .

constraints been addressed in [5]. Thiemann in [4] illustrates an algebraic solution of G_a involving an alternate parametrization of the basic variables. However, the result is to convert H_i constraint into a differential equation in this parametrization, hence presenting the same dilemma albeit in a different form which would seem to imply a kind of duality between gauge transformations and diffeomorphisms. Other approaches to G_a include [7], though the Hamiltonian constraint remains unresolved.

Our proposal for implementing the initial value constraints uses an instanton representation of Plebanski gravity as a starting point. The initial value constraints in the instanton representation are given by

$$H = (\det B)^{1/2} \sqrt{\det \Psi} \left(\frac{1}{2} \left((\text{tr} \Psi)^2 - \text{tr} \Psi^2 \right) + \Lambda \det \Psi \right) = 0 \quad (2)$$

for the Hamiltonian constraint, where Λ is the cosmological constant. The diffeomorphism constraint is given by

$$H_i = \epsilon_{ijk} B_a^j B_e^k \Psi_{ae} = 0, \quad (3)$$

where B_a^i is the magnetic field derived from the Ashtekar connection A_i^a , given by

$$B_a^i = \epsilon^{ijk} \partial_j A_k^a + \frac{1}{2} \epsilon^{ijk} f^{abc} A_j^b A_k^c. \quad (4)$$

The Gauss' law constraint is given by

$$G_a = \mathbf{v}_e \{ \Psi_{ae} \} + C_a^{fg} \Psi_{fg}, \quad (5)$$

where $\mathbf{v}_e = B_e^i \partial_i$ is a triple of vector fields constructed from B_e^i , and

$$C_a^{fg} = (f_{abf} \delta_{ge} + f_{bge} \delta_{af}) C_{be} \quad (6)$$

where $C_{be} = A_i^b B_e^i$ is the so-called magnetic helicity density matrix.² The vector fields \mathbf{v}_a satisfy the commutator bracket

$$[\mathbf{v}_a, \mathbf{v}_b] = h_{ab}^e \mathbf{v}_e, \quad (7)$$

where the structure functions are given by

²See Paper VI regarding the background and physical interpretation behind the magnetic helicity density matrix.

$$h_{ab}^e = (B^{-1})_j^e (B_a^i \partial_i B_b^j - B_b^i \partial_i B_a^j) = (B^{-1})_j^e \mathbf{v}_{[a} \{B_{b]}^j\}. \quad (8)$$

The constraints (2), (3) and (5) can be transformed into the Ashtekar variables for nondegenerate B_a^i and nondegenerate Ψ_{ae} using (1). The present paper will be concerned mainly with the Gauss' law constraint G_a . To obtain the initial value constraints in the Ashtekar variables, one could substitute the inverted form of the CDJ Ansatz

$$\Psi_{ae}^{-1} = B_e^i (\tilde{\sigma}^{-1})_i^a \quad (9)$$

into (2), (3) and (5). This equivalence holds only when the variables are nondegenerate.

The organization of this paper is as follows. In the present section we have written the initial value constraints of GR on (Ψ_{ae}, A_i^a) , the phase space of the instanton representation. The constraints in and of themselves are ultimately constraints on Ψ_{ae} , the momentum space variables of this representation. Section 2 expresses the Gauss' law constraint in operator-matrix form as a triple of differential equations on Ψ_{ae} . These equations are presented in coordinate-invariant form using vector fields $\mathbf{v}_a = B_a^i \partial_i$ tangent to $\tilde{\gamma}$, the congruence of their integral curves chosen to fill 3-space Σ . We present a procedure which inverts the matrix of these vector fields, taking their non-commutativity into account. The result is a propagator from the diagonal (anisotropy) to the off-diagonal (shear) elements of Ψ_{ae} , which constitutes a reduction in three degrees of freedom. Section 3 provides arguments for the invertibility of these vector fields \mathbf{v}_a from different perspectives, based on the chosen configuration A_i^a . In section 4 we provide several examples for various configurations A_i^a where we invert the matrix of vector fields and solve the Gauss' law constraint. Certain configurations are integrable while others are not. By integrability we mean in the sense that the solution is independent of the particular path taken along the integral curves $\tilde{\gamma}$, and depends on the starting and ending points. For the latter nonintegrable configurations, the particular path must be specified in the solution since the solution may be path-dependent. In either case, the algorithm results in a reduction in Ψ_{ae} by three D.O.F. Section 5 treats the inversion of the Gauss' law differential operators for the general case without regard for integrability, and section 6 provides a formal solution to the initial value constraints by expansion about the solution corresponding to the Kodama state.

It will be convenient to regard the basic physical degrees of freedom which are independent of the gauge and of the coordinate system as residing within three elements of Ψ_{ae} , the diagonal elements. The coordinate-dependent, gauge-dependent degrees of freedom are contained within the

connection A_i^a . But it is also convenient to regard these latter degrees of freedom as more fundamentally residing within $\vec{\gamma}$ which is coordinate-invariant. Then the connection and all quantities needed to invert the Gauss' law constraint become derived upon the choice of a coordinate system.

2 Gauss' law constraint in rectangular form

The purpose of the Gauss' law constraint in the instanton representation is, starting from a given configuration A_i^a , to establish a map from three CDJ matrix elements Ψ_{ae} to the remaining six, which constitutes a reduction in three degrees of freedom. Starting from the Gauss' law constraint $\mathbf{v}_e\{\Psi_{ae}\} + C_a^{fg}\Psi_{fg} = 0$, we expand into constituents

$$\begin{aligned} \mathbf{v}_1(\Psi_{11}) + \mathbf{v}_2(\Psi_{12}) + \mathbf{v}_3(\Psi_{13}) + C_1^{fg}\Psi_{fg} &= 0; \\ \mathbf{v}_1(\Psi_{21}) + \mathbf{v}_2(\Psi_{22}) + \mathbf{v}_3(\Psi_{23}) + C_1^{fg}\Psi_{fg} &= 0; \\ \mathbf{v}_1(\Psi_{31}) + \mathbf{v}_2(\Psi_{32}) + \mathbf{v}_3(\Psi_{33}) + C_1^{fg}\Psi_{fg} &= 0, \end{aligned} \quad (10)$$

where we have used the Einstein summation convention. Equation (10) is a set of three linear first-order differential equations in nine unknowns, which implies a reduction of Ψ_{ae} by three degrees of freedom.

It will be necessary to disentangle the CDJ matrix elements in order to assess the nature of the vector spaces being acted upon by the Gauss' law differential operators. Expanding (10) to first disentangle the diagonal (anisotropy) CDJ matrix elements, we have

$$\begin{aligned} \left(\mathbf{v}_1(\Psi_{11}) + C_1^{11}\Psi_{11}\right) + \mathbf{v}_2(\Psi_{12}) + \mathbf{v}_3(\Psi_{13}) + \sum_{f,g \neq 1,1} C_1^{fg}\Psi_{fg} &= 0; \\ \mathbf{v}_1(\Psi_{21}) + \left(\mathbf{v}_2(\Psi_{22}) + C_2^{22}\Psi_{22}\right) + \mathbf{v}_3(\Psi_{23}) + \sum_{f,g \neq 2,2} C_2^{fg}\Psi_{fg} &= 0; \\ \mathbf{v}_1(\Psi_{31}) + \mathbf{v}_2(\Psi_{32}) + \left(\mathbf{v}_3(\Psi_{33}) + C_3^{33}\Psi_{33}\right) + \sum_{f,g \neq 3,3} C_3^{fg}\Psi_{fg} &= 0. \end{aligned} \quad (11)$$

The full expansion of (10) reads

$$\begin{aligned} &\left(\mathbf{v}_1(\Psi_{11}) + C_1^{11}\Psi_{11}\right) + C_1^{22}\Psi_{22} + C_1^{33}\Psi_{33} \\ &+ \left[C_1^{21}\Psi_{21} + C_1^{31}\Psi_{31} + C_1^{23}\Psi_{23} + C_1^{32}\Psi_{32}\right] \\ &+ \left(\mathbf{v}_2(\Psi_{12}) + C_1^{12}\Psi_{12}\right) + \left(\mathbf{v}_3(\Psi_{13}) + C_1^{13}\Psi_{13}\right) = 0 \end{aligned} \quad (12)$$

for the first equation,

$$\begin{aligned} &\left(\mathbf{v}_1(\Psi_{21}) + C_2^{21}\Psi_{21}\right) + \left(\mathbf{v}_2(\Psi_{22}) + C_2^{22}\Psi_{22}\right) + C_2^{11}\Psi_{11} + C_2^{33}\Psi_{33} \\ &+ \left[C_2^{12}\Psi_{12} + C_2^{32}\Psi_{32} + C_2^{13}\Psi_{13} + C_2^{31}\Psi_{31}\right] + \left(\mathbf{v}_3(\Psi_{23}) + C_2^{23}\Psi_{23}\right) = 0 \end{aligned} \quad (13)$$

for the second equation, and

$$\begin{aligned} & \left(\mathbf{v}_1(\Psi_{31}) + C_3^{31}\Psi_{31} \right) + \left(\mathbf{v}_2(\Psi_{32}) + C_3^{32}\Psi_{32} \right) + \left(\mathbf{v}_3(\Psi_{33}) + C_3^{33}\Psi_{33} \right) \\ & + C_3^{11}\Psi_{11} + C_3^{22}\Psi_{22} + \left[C_3^{12}\Psi_{12} + C_3^{13}\Psi_{13} + C_3^{23}\Psi_{23} + C_3^{21}\Psi_{21} \right] = 0 \quad (14) \end{aligned}$$

for the third equation. This system of equations can be more conveniently put into matrix form

$$\begin{aligned} & \begin{pmatrix} (\mathbf{v}_1 + C_1^{11}) & C_1^{22} & C_1^{33} \\ C_2^{11} & (\mathbf{v}_2 + C_2^{22}) & C_2^{33} \\ C_3^{11} & C_3^{22} & (\mathbf{v}_3 + C_3^{33}) \end{pmatrix} \begin{pmatrix} \Psi_{11} \\ \Psi_{22} \\ \Psi_{33} \end{pmatrix} + \\ & \begin{pmatrix} (\mathbf{v}_2 + C_1^{12}) & C_1^{23} & C_1^{31} \\ C_2^{12} & (\mathbf{v}_3 + C_2^{23}) & C_2^{31} \\ C_3^{12} & C_3^{23} & (\mathbf{v}_1 + C_3^{31}) \end{pmatrix} \begin{pmatrix} \Psi_{12} \\ \Psi_{23} \\ \Psi_{31} \end{pmatrix} + \\ & \begin{pmatrix} C_1^{21} & C_1^{32} & (\mathbf{v}_3 + C_1^{13}) \\ (\mathbf{v}_1 + C_2^{21}) & C_2^{32} & C_2^{13} \\ C_3^{21} & (\mathbf{v}_2 + C_3^{32}) & C_3^{13} \end{pmatrix} \begin{pmatrix} \Psi_{21} \\ \Psi_{32} \\ \Psi_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

We next expand the matrix equation into diagonal and off-diagonal parts, using $\Psi_{ae} = \Psi_{(ae)} + \Psi_{[ae]}$. This yields, for the parts involving vector fields,

$$\begin{aligned} & \begin{pmatrix} \mathbf{v}_1 & 0 & 0 \\ 0 & \mathbf{v}_2 & 0 \\ 0 & 0 & \mathbf{v}_3 \end{pmatrix} \begin{pmatrix} \Psi_{11} \\ \Psi_{22} \\ \Psi_{33} \end{pmatrix} + \begin{pmatrix} \mathbf{v}_2 & 0 & \mathbf{v}_3 \\ \mathbf{v}_1 & \mathbf{v}_3 & 0 \\ 0 & \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} \begin{pmatrix} \Psi_{(12)} \\ \Psi_{(23)} \\ \Psi_{(31)} \end{pmatrix} \\ & + \begin{pmatrix} \mathbf{v}_2 & 0 & -\mathbf{v}_3 \\ -\mathbf{v}_1 & \mathbf{v}_3 & 0 \\ 0 & -\mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} \begin{pmatrix} \Psi_{[12]} \\ \Psi_{[23]} \\ \Psi_{[31]} \end{pmatrix} = 0 \end{aligned}$$

2.1 Transformation into Cartesian basis

It will be useful to transform the Gauss' law constraint into a basis of irreducible parts of $SU(2)_- \otimes SU(2)_-$. Let us now decompose the CDJ matrix Ψ_{ae} into orthogonal subspaces consisting of diagonal and off-diagonal parts respectively, given explicitly by

$$\Psi_{ae} = \varphi_f(e^f)_{ae} + \psi_f(\epsilon_f)_{ae} + \Psi_f(E^f)_{ae}. \quad (15)$$

The following identifications have been made

$$\begin{aligned}
\Psi_{(12)} &= \Psi_3; \quad \Psi_{(23)} = \Psi_1; \quad \Psi_{(31)} = \Psi_2; \\
\Psi_{[12]} &= \psi_3; \quad \Psi_{[23]} = \psi_1; \quad \Psi_{[31]} = \psi_2; \\
\Psi_{11} &= \varphi_1; \quad \Psi_{22} = \varphi_2; \quad \Psi_{33} = \varphi_3.
\end{aligned} \tag{16}$$

Equation (15) in matrix form is given by³

$$\Psi_{ae} = \begin{pmatrix} \varphi_1 & \Psi_3 - \psi_3 & \Psi_2 - \psi_2 \\ \Psi_3 + \psi_3 & \varphi_2 & \Psi_1 - \psi_1 \\ \Psi_2 + \psi_2 & \Psi_1 + \psi_1 & \varphi_3 \end{pmatrix}.$$

The basis vectors in this orthogonal decomposition (15) are given by

$$e_{ae}^f = \delta_{af} \delta_{ef}; \quad E_{ae}^f = \sum_{d=1}^3 \delta_d^f |\epsilon_{aed}|, \tag{17}$$

with no summation convention applied to the first definition in (17). These basis elements have the following matrix representation

$$\epsilon_{ae1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \epsilon_{ae2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \epsilon_{ae3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{ae}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad E_{ae}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad E_{ae}^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_{ae}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad e_{ae}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad e_{ae}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which satisfy the orthogonality relations

$$\begin{aligned}
\langle e^f | e^g \rangle &= \sum_{a,e} (e^f)_{ae} (e^g)^{ea} = \delta^{fg}; \\
\langle E^\alpha | E^\beta \rangle &= \sum_{a,e} E_{ae}^\alpha E^{\beta ea} = 2\delta^{\alpha\beta}; \\
\langle E^\alpha | e^f \rangle &= \sum_{a,e} E_{ae}^\alpha (e^f)^{ea} = 0.
\end{aligned} \tag{18}$$

³The diagonal parts φ_f have the interpretation of anisotropy components, while the off-diagonal symmetric Ψ_f and antisymmetric ψ_f components have the interpretation respectively of shear and rotation components.

with corresponding completeness relations

$$\sum \epsilon_{dae} \epsilon_{dbf} + \sum_d (e_d)^{ae} (e^d)_{bf} + \sum_\alpha E_{ae}^\alpha E_\alpha^{bf} = \delta_a^b \delta_e^f. \quad (19)$$

In the new basis, the Gauss' law constraint can then be written in the compact form

$$M_{af} \Psi_f + V_{af} \psi_f + H_{af} \varphi_f = 0, \quad (20)$$

where V , M and H are Gauss' law operators with respect to the shear, rotation and anisotropy subspaces, given by

$$M_{ae} = E_{ae}^f \mathbf{w}_f; \quad V_{ae} = \epsilon_{ae}^f \mathbf{w}_f; \quad H_{ae} = e_{ae}^f \mathbf{w}_f. \quad (21)$$

The part of the constraint depending purely upon vector fields is given by

$$\begin{aligned} \mathbf{v}_e \{ \Psi_{ae} \} &= \begin{pmatrix} \mathbf{v}_2 & 0 & -\mathbf{v}_3 \\ -\mathbf{v}_1 & \mathbf{v}_3 & 0 \\ 0 & -\mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} \begin{pmatrix} \psi_3 \\ \psi_1 \\ \psi_2 \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{v}_2 & 0 & \mathbf{v}_3 \\ \mathbf{v}_1 & \mathbf{v}_3 & 0 \\ 0 & \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} + \begin{pmatrix} \mathbf{v}_1 & 0 & 0 \\ 0 & \mathbf{v}_2 & 0 \\ 0 & 0 & \mathbf{v}_3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \end{aligned}$$

and the part depending upon the helicity density matrix C_{be} by⁴

$$\begin{aligned} C_a^{fg} \Psi_{fg} &= \begin{pmatrix} C_{13} & -(C_{33} + C_{22}) & C_{12} \\ C_{23} & C_{12} & -(C_{11} + C_{33}) \\ -(C_{22} + C_{11}) & C_{31} & C_{32} \end{pmatrix} \begin{pmatrix} \psi_3 \\ \psi_1 \\ \psi_2 \end{pmatrix} \\ &+ \begin{pmatrix} C_{13} - 2C_{31} & C_{22} - C_{33} & 2C_{21} - C_{12} \\ 2C_{32} - C_{23} & C_{21} - 2C_{12} & C_{33} - C_{11} \\ C_{11} - C_{22} & 2C_{13} - C_{31} & C_{32} - 2C_{23} \end{pmatrix} \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} \\ &+ \begin{pmatrix} C_{32} - C_{23} & -C_{32} & C_{23} \\ C_{31} & C_{13} - C_{31} & -C_{13} \\ -C_{21} & C_{12} & C_{21} - C_{12} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \end{aligned}$$

The vector field contributions are matrices whose elements consist of non-commuting differential operators. We will need to invert these matrices in order to implement the Gauss' law constraint.

⁴See Appendix A for the derivation of the individual terms.

2.2 Formal inversion in the Cartesian representation

From the diffeomorphism constraint (3), the antisymmetric part of the CDJ matrix vanishes $\psi_d = \epsilon_{dae} \Psi_{ae} = 0$, and the Gauss' law constraint reduces to just the symmetric part

$$H_{ae} \varphi_e + M_{ae} \Psi_e = 0. \quad (22)$$

Equation (22) is a linear relation between the shear elements Ψ_e and the anisotropy elements φ_e . We can reduce Ψ_{ae} by three additional D.O.F. by solving for Ψ_e in terms of φ_e , which in essence establishes a map $\vec{\varphi} \rightarrow \vec{\Psi}$. The constraint can be written in the general form

$$(\mathbf{v}_{ae} + \alpha_{ae}) \Psi_e = Q_a[\vec{\varphi}] \longrightarrow \Psi_e, \quad (23)$$

where we have defined the operator matrix of vector fields on the shear subspace

$$\mathbf{v}_{ae} = \begin{pmatrix} \mathbf{v}_2 & 0 & \mathbf{v}_3 \\ \mathbf{v}_1 & \mathbf{v}_3 & 0 \\ 0 & \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix},$$

with the helicity density matrix insertions

$$\alpha_{ae} = \begin{pmatrix} C_{13} - 2C_{31} & C_{22} - C_{33} & 2C_{21} - C_{12} \\ 2C_{32} - C_{23} & C_{21} - 2C_{12} & C_{33} - C_{11} \\ C_{11} - C_{22} & 2C_{13} - C_{31} & C_{32} - 2C_{23} \end{pmatrix},$$

and on the anisotropy subspace by

$$Q_a[\vec{\varphi}] = - \begin{pmatrix} \mathbf{v}_1 - C_{[23]} & -C_{32} & C_{23} \\ C_{31} & \mathbf{v}_2 - C_{[31]} & -C_{13} \\ -C_{21} & C_{12} & \mathbf{v}_3 - C_{[12]} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

where $C_{[ab]} = C_{ab} - C_{ba}$. The formal solution to (23) is given by

$$(\mathbf{v}_{ea} + \alpha_{ea})^{-1} Q_a[\vec{\varphi}], \quad (24)$$

which may be written as an infinite operator expansion,

$$\Psi_e = \left[\delta_{ee_0} \left(\sum_{n=0}^{\infty} (-1)^n \mathbf{v}_{e_0 e_1}^{-1} \alpha_{e_1 e_2} \mathbf{v}_{e_2 e_3}^{-1} \alpha_{e_3 e_4} \cdots \mathbf{v}_{e_{n-1} e_n}^{-1} \right) \delta_{e_n a} \right] \mathbf{v}_{af}^{-1} H_{fg} \varphi_g \equiv \hat{J}_e^g \varphi_g, \quad (25)$$

where \mathbf{v}_{ae}^{-1} acts to the right. We have defined the Gauss' law propagator by the operator $\hat{J}_f^g = -(M^{-1}H)_f^d$, where M^{-1} must be suitably defined. First let us carry out the inversion of the part involving just vector fields.

We would like to express the shear elements $\vec{\Psi}$ as the image of the anisotropy elements $\vec{\varphi}$ under the map \hat{J}_f^g . If one chooses \mathbf{v}_{ae} to consist entirely of vector fields with the helicity density contained in α_{ae} , then in order to solve G_a one must invert the matrix \mathbf{v}_{ae} . More precisely, we must find an operator-valued matrix \mathbf{u}_{ae} such that $\mathbf{u}_{ae}\mathbf{v}_{ef} = \delta_{ef}$ is the identity operator. Since \mathbf{v}_{ae} consists of noncommuting vector fields \mathbf{v}_a , then care must be taken with operator ordering when carrying out the inversion. Let the required matrix be given by

$$\mathbf{u}_{ae} = \delta_{a1}a_e + \delta_{a2}b_e + \delta_{a3}c_e, \quad (26)$$

where a_e , b_e and c_e are operators which remain to be determined. The resulting operator product $\mathbf{u}_{ae}\mathbf{v}_{ef}$ is given by

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} \mathbf{v}_2 & 0 & \mathbf{v}_3 \\ \mathbf{v}_1 & \mathbf{v}_3 & 0 \\ 0 & \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} = \begin{pmatrix} a_1\mathbf{v}_2 + a_2\mathbf{v}_1 & a_2\mathbf{v}_3 + a_3\mathbf{v}_2 & a_1\mathbf{v}_3 + a_3\mathbf{v}_1 \\ b_1\mathbf{v}_2 + b_2\mathbf{v}_1 & b_2\mathbf{v}_3 + b_3\mathbf{v}_2 & b_1\mathbf{v}_3 + b_3\mathbf{v}_1 \\ c_1\mathbf{v}_2 + c_2\mathbf{v}_1 & c_2\mathbf{v}_3 + c_3\mathbf{v}_2 & c_1\mathbf{v}_3 + c_3\mathbf{v}_1 \end{pmatrix}.$$

We must now choose (a_e, b_e, c_e) to diagonalize the matrix product, preserving the operator ordering.

First we require that the off-diagonal elements vanish. Starting with the first row of $\mathbf{u}_{ae}\mathbf{v}_{ef}$, this leads to the equations

$$\begin{aligned} a_1\mathbf{v}_3 + a_3\mathbf{v}_1 &= 0 \longrightarrow a_3 = -a_1\mathbf{v}_3\mathbf{v}_1^{-1}; \\ a_2\mathbf{v}_3 + a_3\mathbf{v}_2 &= 0 \longrightarrow a_2 = -a_3\mathbf{v}_2\mathbf{v}_3^{-1} = a_1\mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2\mathbf{v}_3^{-1}. \end{aligned} \quad (27)$$

The requirement that the off-diagonal elements of $\mathbf{u}_{ae}\mathbf{v}_{ef}$ in the second row vanish is given by

$$\begin{aligned} b_1\mathbf{v}_2 + b_2\mathbf{v}_1 &= 0 \longrightarrow b_1 = -b_2\mathbf{v}_1\mathbf{v}_2^{-1}; \\ b_1\mathbf{v}_3 + b_3\mathbf{v}_1 &= 0 \longrightarrow b_3 = -b_1\mathbf{v}_3\mathbf{v}_1^{-1} = b_2\mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3\mathbf{v}_1^{-1}. \end{aligned} \quad (28)$$

The requirement that the off-diagonal elements of $\mathbf{u}_{ae}\mathbf{v}_{ef}$ in the third row vanish is given by

$$\begin{aligned} c_2\mathbf{v}_3 + c_3\mathbf{v}_2 &= 0 \longrightarrow c_2 = -c_3\mathbf{v}_2\mathbf{v}_3^{-1}; \\ c_1\mathbf{v}_2 + c_2\mathbf{v}_1 &= 0 \longrightarrow c_1 = -c_2\mathbf{v}_1\mathbf{v}_2^{-1} = c_3\mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1\mathbf{v}_2^{-1}. \end{aligned} \quad (29)$$

Having required all off-diagonal parts of $\mathbf{u}_{ae}\mathbf{v}_{ef}$ to vanish we can compute the diagonal parts using (27), (28) and (29). These are given by

$$\begin{aligned} a_1\mathbf{v}_2 + a_2\mathbf{v}_1 &= a_1(\mathbf{v}_2 + \mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1); \\ b_2\mathbf{v}_3 + b_3\mathbf{v}_2 &= b_2(\mathbf{v}_3 + \mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2); \\ c_1\mathbf{v}_3 + c_3\mathbf{v}_1 &= c_3(\mathbf{v}_1 + \mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3). \end{aligned} \quad (30)$$

Hence, three elements of \mathbf{u}_{ae} , namely a_1 , b_2 and c_e remain undetermined. However, the matrix product of $\mathbf{u}_{ae}\mathbf{v}_{ef}$ can now be written in the form

$$\begin{aligned} &\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2\mathbf{v}_3^{-1} & -\mathbf{v}_3\mathbf{v}_1^{-1} \\ -\mathbf{v}_1\mathbf{v}_2^{-1} & 1 & \mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3\mathbf{v}_1^{-1} \\ \mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1\mathbf{v}_2^{-1} & -\mathbf{v}_2\mathbf{v}_3^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_2 & 0 & \mathbf{v}_3 \\ \mathbf{v}_1 & \mathbf{v}_3 & 0 \\ 0 & \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} \mathbf{v}_2 + \mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1 & 0 & 0 \\ 0 & \mathbf{v}_3 + \mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2 & 0 \\ 0 & 0 & \mathbf{v}_1 + \mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3 \end{pmatrix}. \end{aligned}$$

The undetermined operators appear to the left on both sides. Assuming invertibility of these operators we act with $Diag(a_1^{-1}, b_2^{-1}, c_3^{-1})$ on both sides to obtain the following decomposition

$$\mathbf{u}_{ae} = \mathbf{v}_{ae}^{-1} = \begin{pmatrix} \mathbf{v}_2 & 0 & \mathbf{v}_3 \\ \mathbf{v}_1 & \mathbf{v}_3 & 0 \\ 0 & \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix}^{-1} = (1 + \pi)^{-1}\mathbf{v}^{-1}M,$$

where we have defined

$$M = \begin{pmatrix} 1 & \mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2\mathbf{v}_3^{-1} & -\mathbf{v}_3\mathbf{v}_1^{-1} \\ -\mathbf{v}_1\mathbf{v}_2^{-1} & 1 & \mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3\mathbf{v}_1^{-1} \\ \mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1\mathbf{v}_2^{-1} & -\mathbf{v}_2\mathbf{v}_3^{-1} & 1 \end{pmatrix}$$

and

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_2 & 0 & 0 \\ 0 & \mathbf{v}_3 & 0 \\ 0 & 0 & \mathbf{v}_1 \end{pmatrix}; \quad \pi = \begin{pmatrix} \pi_2 & 0 & 0 \\ 0 & \pi_3 & 0 \\ 0 & 0 & \pi_1 \end{pmatrix}$$

as well as the cyclic operators

$$\pi_2 = \mathbf{v}_2^{-1}\mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1; \quad \pi_3 = \mathbf{v}_3^{-1}\mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3\mathbf{v}_1^{-1}\mathbf{v}_2; \quad \pi_1 = \mathbf{v}_1^{-1}\mathbf{v}_2\mathbf{v}_3^{-1}\mathbf{v}_1\mathbf{v}_2^{-1}\mathbf{v}_3. \quad (31)$$

One could then in principle evaluate (25) explicitly term by term, starting from a given configuration A_i^a .

Solving the Gauss' law constraint should reduce Ψ_{ae} by three degrees of freedom to the anisotropy subspace φ_f , given by

$$\Psi_{ae} = (e_{ae}^g - E_{ae}^f \hat{J}_f^g[\vec{A}]) \varphi_g \equiv T_{ae}^g \varphi_g. \quad (32)$$

The physical interpretation is that upon projection to the Gauss' law constraint surface, the anisotropy basis elements e_{ae}^f become distorted into T_{ae}^f based upon the configuration A_i^a , in matrix form

$$T_{ae}^1 = \begin{pmatrix} 1 & -\hat{J}_3^1 & -\hat{J}_2^1 \\ -\hat{J}_3^1 & 0 & -\hat{J}_1^1 \\ -\hat{J}_1^2 & -\hat{J}_1^1 & 0 \end{pmatrix}; \quad T_{ae}^2 = \begin{pmatrix} 0 & -\hat{J}_3^2 & -\hat{J}_2^2 \\ -\hat{J}_3^2 & 1 & -\hat{J}_1^2 \\ -\hat{J}_2^2 & -\hat{J}_1^2 & 0 \end{pmatrix}; \quad T_{ae}^3 = \begin{pmatrix} 0 & -\hat{J}_3^3 & -\hat{J}_2^3 \\ -\hat{J}_3^3 & 0 & -\hat{J}_1^3 \\ -\hat{J}_2^3 & -\hat{J}_1^3 & 1 \end{pmatrix}.$$

The shear elements are eliminated and φ_f are now expressed in a 'curved' basis $T_{ae}^f = T_{ae}^f[\vec{A}]$ which encodes the kinematic effects of the Gauss' law constraint for each configuration A_i^a .⁵

2.3 Off-diagonal helicity configurations

Another possibility to carry out the inversion is to define the insertion matrix α_{ae} by

$$\alpha_{ae} = \begin{pmatrix} 0 & C_{22} - C_{33} & 0 \\ 0 & 0 & C_{33} - C_{11} \\ C_{11} - C_{22} & 0 & 0 \end{pmatrix}.$$

which leaves the remaining matrix containing vector fields

$$\mathbf{w}_{ae} = \begin{pmatrix} \mathbf{v}_2 + C_{13} - 2C_{31} & 0 & \mathbf{v}_3 + 2C_{21} - C_{12} \\ \mathbf{v}_1 + 2C_{32} - C_{23} & \mathbf{v}_3 + C_{21} - 2C_{12} & 0 \\ 0 & \mathbf{v}_2 + 2C_{13} - C_{31} & \mathbf{v}_1 + C_{32} - 2C_{23} \end{pmatrix}$$

The solution to the Gauss' law constraint then amounts to carrying out the analogue of (25), namely

$$\Psi_e = \left[\delta_{ee_0} \left(\sum_{n=0}^{\infty} (-1)^n \mathbf{w}_{e_0 e_1}^{-1} \alpha_{e_1 e_2} \mathbf{w}_{e_2 e_3}^{-1} \alpha_{e_3 e_4} \dots \mathbf{w}_{e_{n-1} e_n}^{-1} \right) \delta_{e_n a} \right] \mathbf{w}_{af}^{-1} H_{fg} \varphi_g \equiv \hat{J}_e^g \varphi_g, \quad (33)$$

⁵It will be convenient to regard the set of all nonorthogonal charts of Σ as being the fundamental structure. Then the magnetic field B_e^i becomes derived from the vector fields tangent to the three linearly independent coordinate directions defining integral curves through each point. Hence, there is a correspondence from T_{ae}^f to each chart, where the transformation from one chart to another induces a passive diffeomorphism.

which in the general case may be more manageable. For off-diagonal helicity configurations, defined as configurations for A_i^a for which $C_{11} = C_{22} = C_{33}$, the inversion can be written in closed form since $\alpha_{ae} = 0$, which reduces equation (33) to just the first term \mathbf{w}_{ae}^{-1} . One then explicitly writes the Gauss' law propagator as $J_f^g = \mathbf{w}_{fe}^{-1} H_{ae}$, where

$$\mathbf{w}_{ae}^{-1} = \begin{pmatrix} \mathbf{w}_2 & 0 & \mathbf{w}_3 \\ \mathbf{w}_1 & \mathbf{w}_3 & 0 \\ 0 & \mathbf{w}_2 & \mathbf{w}_1 \end{pmatrix}^{-1} = (1 + \pi)^{-1} \mathbf{w}^{-1} M,$$

where we have defined

$$M = \begin{pmatrix} 1 & \mathbf{w}_3 \mathbf{w}_1^{-1} \mathbf{w}_2 \mathbf{w}_3^{-1} & -\mathbf{w}_3 \mathbf{w}_1^{-1} \\ -\mathbf{w}_1 \mathbf{w}_2^{-1} & 1 & \mathbf{w}_1 \mathbf{w}_2^{-1} \mathbf{w}_3 \mathbf{w}_1^{-1} \\ \mathbf{w}_2 \mathbf{w}_3^{-1} \mathbf{w}_1 \mathbf{w}_2^{-1} & -\mathbf{w}_2 \mathbf{w}_3^{-1} & 1 \end{pmatrix}$$

and

$$\mathbf{w} = \begin{pmatrix} \mathbf{w}_2 & 0 & 0 \\ 0 & \mathbf{w}_3 & 0 \\ 0 & 0 & \mathbf{w}_1 \end{pmatrix}; \quad \pi = \begin{pmatrix} \pi_2 & 0 & 0 \\ 0 & \pi_3 & 0 \\ 0 & 0 & \pi_1 \end{pmatrix}$$

as well as the cyclic operators⁶

$$\begin{aligned} \pi_2 &= \mathbf{w}_2^{-1} \mathbf{w}_3 \mathbf{w}_1^{-1} \mathbf{w}_2 \mathbf{w}_3^{-1} \mathbf{w}_1; \\ \pi_3 &= \mathbf{w}_3^{-1} \mathbf{w}_1 \mathbf{w}_2^{-1} \mathbf{w}_3 \mathbf{w}_1^{-1} \mathbf{w}_2; \\ \pi_1 &= \mathbf{w}_1^{-1} \mathbf{w}_2 \mathbf{w}_3^{-1} \mathbf{w}_1 \mathbf{w}_2^{-1} \mathbf{w}_3. \end{aligned} \tag{34}$$

One has then reduced the problem to the inversion of individual first order differential operators of the form $\mathbf{w} = \mathbf{v} + c$, where c is a c-number. In this case one may be able to find an integrating factor to construct \mathbf{w}^{-1} . The series (25) formally solves the Gauss' law constraint by construction. However, the solution does not exist until it has shown to be well-defined. We will put in place the ingredients necessary to properly define all terms and their constituent operators, as well as the action within their domains of definition, and provide some examples.

⁶The physical interpretation is best motivated by analogy to the case $\mathbf{v}_a = \partial_a$, where the vector fields are exact partial differentials. Then the vector fields form the structure of an abelian group where differentiation are group elements with inverses $\partial_a^{-1} = \int dx^a$, which corresponds to antidifferentiation with vanishing boundary data. The cyclic operators correspond to making one complete closed path along the edges of a hexagon in group space. For the aforementioned case the cyclic operators would act as the identity element on the space of functions. In the general case the cyclic operators are not the identity, owing to the noncommutativity of the vector fields.

3 Hamiltonian flow and integral curves

A necessary condition for the well-definedness of the expansion (25) hinges upon the invertibility of the vector fields $\mathbf{v}_a = B_a^i \partial_i = X_a + N_a$, where we have defined

$$X_a = \epsilon^{ijk} (\partial_j A_k^a) \partial_i; \quad N_a = \frac{1}{2} \epsilon^{ijk} f_{abc} (A_j^b A_k^c) \partial_i = (\det A) (A^{-1})_a^i \partial_i \quad (35)$$

respectively as the abelian and nonabelian contributions for $\det A \neq 0$. The abelian part of \mathbf{v}_a is given by⁷

$$X_a = 2 \left[\left(\frac{\partial A_1^a}{\partial y} - \frac{\partial A_2^a}{\partial x} \right) \frac{\partial}{\partial z} + \left(\frac{\partial A_2^a}{\partial z} - \frac{\partial A_3^a}{\partial y} \right) \frac{\partial}{\partial x} + \left(\frac{\partial A_3^a}{\partial x} - \frac{\partial A_1^a}{\partial z} \right) \frac{\partial}{\partial y} \right]. \quad (36)$$

Rearranging the terms of (36), one obtains

$$X_a = X_{A_1^a} + X_{A_2^a} + X_{A_3^a} \quad (37)$$

where we have defined

$$\begin{aligned} X_{A_1^a} &= \left(\frac{\partial A_1^a}{\partial y} \right) \frac{\partial}{\partial z} - \left(\frac{\partial A_1^a}{\partial z} \right) \frac{\partial}{\partial y}; \\ X_{A_2^a} &= \left(\frac{\partial A_2^a}{\partial z} \right) \frac{\partial}{\partial x} - \left(\frac{\partial A_2^a}{\partial x} \right) \frac{\partial}{\partial z}; \\ X_{A_3^a} &= \left(\frac{\partial A_3^a}{\partial x} \right) \frac{\partial}{\partial y} - \left(\frac{\partial A_3^a}{\partial y} \right) \frac{\partial}{\partial x}. \end{aligned} \quad (38)$$

Each term of (37) exhibits the same form as a Hamiltonian vector field X_H from classical mechanics, given by

$$X_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p}. \quad (39)$$

If one defines a symplectic manifold on phase space $\Omega = (p, q)$, with symplectic two form $\omega = dp \wedge dq$, then the corresponding Hamiltonian equations of motion would be given by

$$\dot{q} = \frac{\delta H}{\delta p}; \quad \dot{p} = -\frac{\delta H}{\delta q}. \quad (40)$$

⁷We have chosen x, y and z as labels for the coordinates, which needn't be Cartesian. These symbols can be used for arbitrary 3-dimensional coordinates.

The Hamiltonian function H generates evolution in parameter t along a curve in Ω , and Ω is foliated by the orbits of the Hamiltonian flow such that a unique curve passes through each point.

Hence, identify with the components of the connection A_i^a nine Hamiltonian functions A_1^a , A_2^a and A_3^a on respective phase spaces (y, z) , (z, x) and (x, y) . These imply Hamilton's equations of motion

$$\frac{dx^j}{dt_i^a} = \{x^j, A_i^a\}; \quad \frac{dx^k}{dt_i^a} = \{x^k, A_i^a\}, \quad (41)$$

with cyclic permutations over i, j, k . Hence, for each Hamiltonian function A_i^a there exists a parameter t_i^a generating flow along a curve in the two dimensional (x^j, x^k) hyperplanes dual to each corresponding i direction. We would like to envision the Hamiltonian flows as the projection of integral curves of X_a onto these hyperplanes. To see that these curves are well-defined, re-arrange X_a into the following form

$$X_a = - \sum_{i=1}^3 I_{ijk} \left(\frac{\partial A_j^a}{\partial x^k} - \frac{\partial A_k^a}{\partial x^j} \right) \frac{\partial}{\partial x^i}. \quad (42)$$

The three dimensional curves ϕ for each i, j, k imply the relations

$$\frac{\partial A_i^a}{\partial x^j} = \frac{dx^k}{dt_i^a}; \quad \frac{\partial A_i^a}{\partial x^k} = - \frac{dx^j}{dt_i^a}. \quad (43)$$

The total variation of A_i^a in 3-space is given by

$$dA_i^a = \frac{\partial A_i^a}{\partial x^i} dx^i + \frac{\partial A_i^a}{\partial x^j} dx^j + \frac{\partial A_i^a}{\partial x^k} dx^k = \frac{\partial A_i^a}{\partial x^i} dx^i. \quad (44)$$

Hence when restricted to the curve parametrized by t_i^a , then A_i^a is constant, which is consistent with the 2-dimensional projections implied by (41). The vector field X_a can be seen as the composition of all three curves

$$(\mathbf{v}_a)_{Abelian} = \frac{d}{dt_1^a} + \frac{d}{dt_2^a} + \frac{d}{dt_3^a} = X_a \equiv \frac{d}{dt^a}. \quad (45)$$

If the individual contributions could be exponentiated, then (45) has the following action on an arbitrary function f

$$(\mathbf{v}_a)_{Abelian} \{f\} = \frac{d}{dt} f \left(e^{tX_{A_1^a}} e^{tX_{A_2^a}} e^{tX_{A_3^a}} x \right) \Big|_{t=0}, \quad (46)$$

which ascribes to all three vector fields a flow labelled by the same parameter t . The order of the exponentials in (46) does not matter, since they commute for $t = 0$. Hence (45) illustrates the fact that the vector fields form a vector space. Since this gives a well-defined flow, then the inverse of the abelian contribution $(\mathbf{v}_a^{-1})_{Abelian} \equiv X_a^{-1}$ exists and admits an interpretation of integration along the integral curves ϕ .

3.1 Differential geometry of differential forms

The integrability of the abelian part of \mathbf{v}_a can be expressed more succinctly using differential forms. This corresponds to three vector potential one-forms $A^a = A_i^a dx^i$, with field strength given by

$$F^a = dA^a + \frac{1}{2} f^{abc} A^b \wedge A^c. \quad (47)$$

The vector field X_a from (35) can be derived from the two form dA^a in the following manner. Define η , the top three form on 3-space Σ by

$$\eta = \frac{1}{3} \epsilon_{lmn} dx^l \wedge dx^m \wedge dx^n. \quad (48)$$

The interior product of (48) with X_a is given by

$$\begin{aligned} i_{X_a}(\eta) &= \langle \eta, X_a \rangle = \frac{1}{3} \epsilon^{ijk} \epsilon_{lmn} \partial_i A_j^a \langle \partial_k, dx^l \wedge dx^m \wedge dx^n \rangle \\ &= \frac{1}{3} \epsilon^{ijk} \epsilon_{lmn} \partial_i A_j^a (\delta_k^l dx^m \wedge dx^n + \delta_k^m dx^n \wedge dx^l + \delta_k^n dx^l \wedge dx^m). \end{aligned} \quad (49)$$

All three terms on the right hand side of (49) are equal, summing to the abelian part of the field strength, which leads to the relation

$$i_{X_a}(\eta) = dA^a \longrightarrow d(i_{X_a}(\eta)) = d^2 A^a = 0. \quad (50)$$

Therefore, if X_a is integrable, then this must be related to the fact that $d^2 A^a = 0$. But the vector fields $\mathbf{v}_a = X_a + N_a$ also have a contribution from the nonabelian part N_a . Therefore we can rephrase the criterion for integrability of \mathbf{v}_a as a condition on N_a , namely that

$$d(i_{N_a}(\eta)) = d(i_{X_a}(\eta)) = 0. \quad (51)$$

Using (47) this leads to a sufficient condition for integrability of \mathbf{v}_a that

$$f^{abc} A^b \wedge dA^c = \frac{1}{2} d(f^{abc} A^b \wedge A^c) = 0 \longrightarrow \frac{1}{2} f^{abc} A^b \wedge A^c = d\Phi^a \quad (52)$$

locally for some one form $\Phi^a = \Phi_i^a dx^i$.

There is a wide class of configurations A^a for which this is the case. For example, for potentials of the form $A^a = d\phi^a$ for $SO(3, C)$ -valued scalars ϕ_a we have that $X_a = 0$, and N_a in component form is given by

$$\mathbf{v}_a = N_a = \frac{1}{2} f^{abc} \epsilon^{ijk} (\partial_j \phi^a) (\partial_k \phi^b) \partial_i. \quad (53)$$

By the above results on differential forms \mathbf{v}_a is integrable since $d(f^{abc} \phi^b \wedge d\phi^c) = 0$, which means that \mathbf{v}_a^{-1} exists. Moreover, the helicity density matrix

$$C_{ae} = A_i^a B_e^i = \frac{1}{2} f^{abc} \epsilon^{ijk} (\partial_j \phi^b) (\partial_k \phi^c) (\partial_i \phi^e) = \delta_{ae} (\det \partial \vec{\phi}) \quad (54)$$

is diagonal. Hence the c-number insertion matrix α_{ae} vanishes and only the first term of (25), \mathbf{v}_{ae}^{-1} , survives. The result is that for the configuration $A_i^a = \partial_i \phi^a$, where $\phi^a = (\phi^1, \phi^2, \phi^3)$ are arbitrary functions in Σ , the Gauss' law constraint should have an explicit solution.

Equation (51) has the following interpretation, from Cartan's formula for differential forms [9]

$$L_{X_a} \eta = i_{X_a} (d\eta) + d(i_{X_a} \eta). \quad (55)$$

Using the identity $d\eta = 0$, this states that the Lie derivative of the top 3-form on a 3-dimensional space along the direction of X_a must vanish.

In the more general case where $A^b \wedge A^c \neq d(f^{abc} \phi^c)$ is not exact, one should still attempt to verify the integrability of G_a . The issue of integrability in relation to the most general magnetic field can best be understood from the vantage point of the Bianchi identity,

$$dF^a + f^{abc} A^b \wedge F^c = 0, \quad (56)$$

which we posit is directly related to the vector fields \mathbf{v}_a . The following relation ensues, in direct analogy to (50)

$$i_{\mathbf{v}_a}(\eta) = F^a, \quad (57)$$

where F^a is the total nonabelian field strength. By the Cartan equation we have

$$L_{\mathbf{v}_a}(\eta) = di_{\mathbf{v}_a}(\eta) + i_{\mathbf{v}_a}(d\eta). \quad (58)$$

The second term of (58) vanishes and we are left with

$$L_{\mathbf{v}_a}(\eta) = dF^a = -f^{abc}A^b \wedge dA^c \quad (59)$$

where we have used (56) in conjunction with $d^2A^a = 0$. This can be rewritten as

$$L_{\mathbf{v}_a}(\eta) = -(f^{abc}C_{bc})\eta. \quad (60)$$

Equation (60) implies that a nonvanishing antisymmetric part of the helicity density could possibly present an obstruction to the integrability of G_a .

3.2 The various possibilities

Hence the possibilities can be delineated as follows. If A^a is closed, then $dA^a = 0$, which leads to two subcases according to whether A^a is exact or not exact.

(i) Case(i): A^a closed and exact. In this case one can write $A^a = d\phi^a$ for three arbitrary functions ϕ^a , and one has that $X_a = 0$. Hence the only remaining issue regards N_a , which can be written

$$N_a = f^{abc}d\phi^b \wedge d\phi^c = d(f^{abc}\phi^b d\phi^c) \quad (61)$$

which is exact. For this case one has

$$\mathbf{v}_a = f^{abc}\epsilon_{ijk}\partial_j(\phi^b\partial_k\phi^c)\partial_i, \quad (62)$$

which is integrable. This case for each A^a defines a map between from coordinate system (x, y, z) to (ϕ^1, ϕ^2, ϕ^3) , which defines an equivalence class of passive diffeomorphisms.

Case(ii): A_i^a closed and not exact. In this case X_a is still integrable even if not explicitly so in terms of global coordinates. It is not possible to judge the integrability of N_a based on this, since $f^{abc}A^b \wedge dA^c \neq 0$ in general. So one must examine a less restrictive notion of integrability,

$$(f^{abc}A^b \wedge A^c) \wedge d(f^{afg}A^f \wedge A^g) = 0 \quad (63)$$

with no summation over a . Equation (63) is valid since it corresponds to a 5-form in three dimensional space Σ , which must vanish. The result is that for A^a closed, the Gauss' law constraint should still be integrable.

Case(iii): A_i^a not closed ($dA^a \neq 0$). Still, dA^a is integrable since as we have demonstrated it induces a Hamiltonian flow on 2-dimensional dual hyperplanes. Integrability then hinges upon whether N_a is integrable, which leads to two subcases. First, if $f^{abc}A^b \wedge A^c$ is closed then this implies that

$$d(f^{abc}A^b \wedge A^c) = f^{abc}A^b \wedge dA^c = -(f^{abc}C_{bc}) = 0, \quad (64)$$

or the antisymmetric part of the magnetic helicity $C_{[ae]}$ must vanish. Certainly, as we will demonstrate, there are a wide class of configurations A_i^a for which this is the case.

The second possibility is when $C_{[bc]} \neq 0$, namely that the antisymmetric part of the helicity is nonvanishing with A_i^a not closed. While N_a does not correspond to a closed two form, one can demonstrate integrability using the same criterion as in (63)

$$(f^{abc}A^b \wedge A^c) \wedge d(f^{afg}A^f \wedge A^g) = 0, \quad (65)$$

with no summation over a , which is due to the vanishing of a 5-form in three dimensional space Σ . The result is that for generic configurations A_i^a , the vector fields comprising the Gauss' law constraint \mathbf{v}_a should be integrable.

Case (iv): We have considered a case where $X_a = 0$ and $N_a \neq 0$, which yields an infinite set of integrable configurations labelled by three arbitrary functions ϕ^a . Let us now consider the other extreme, where $N_a = 0$ and $X_a \neq 0$. The nonabelian contributions to \mathbf{v}_a are given in component form by

$$\begin{aligned} N_1 &= (A_2^2 A_3^3 - A_3^2 A_2^3) \partial_x + (A_3^2 A_1^3 - A_1^2 A_3^3) \partial_y + (A_1^2 A_2^3 - A_2^2 A_1^3) \partial_z; \\ N_2 &= (A_3^3 A_1^1 - A_1^3 A_3^1) \partial_y + (A_1^3 A_2^1 - A_2^3 A_1^1) \partial_z + (A_2^3 A_3^1 - A_3^3 A_2^1) \partial_x; \\ N_3 &= (A_1^1 A_2^2 - A_2^1 A_1^2) \partial_z + (A_2^1 A_3^2 - A_3^1 A_2^2) \partial_x + (A_3^1 A_1^2 - A_1^1 A_3^2) \partial_y. \end{aligned} \quad (66)$$

The total contribution is given by

$$\mathbf{v}_a = X_a + \sum_i \text{cof}(A_i^a) \frac{\partial}{\partial x^i}. \quad (67)$$

One possible obstruction to integrability of the Gauss' law constraint is due to $N_a = \text{cof}(A_i^a) \partial_i$, since we have not shown this contribution to correspond to a flow orbit. A sufficient condition for the invertibility of \mathbf{v}_a is that

$N_a = 0 \forall a$. One sees that it is sufficient to require that two rows of cofactors to vanish, which ensures that the third row also vanishes. In other words, for example taking

$$A_2^2 A_3^3 - A_3^2 A_2^3 = A_3^2 A_1^3 - A_1^2 A_3^3 = A_1^2 A_2^3 - A_2^2 A_1^3 = 0, \quad (68)$$

and

$$A_3^3 A_1^1 - A_1^3 A_3^1 = A_1^3 A_2^1 - A_2^3 A_1^1 = A_2^3 A_3^1 - A_3^3 A_2^1 = 0 \quad (69)$$

yields that $N_3 = 0$, with analogous results for the different combinations of rows. The result is that the unconstrained components of A_i^a form a five dimensional manifold, with $9!/5!4! = 126$ possibilities. We provide one particular choice for illustrative purposes

$$A_i^a = \begin{pmatrix} A_1^1 & A_2^1 & A_3^1 \\ A_1^1 A_2^2 (A_2^1)^{-1} & A_2^2 & A_2^2 A_3^1 (A_2^1)^{-1} \\ A_3^3 A_1^1 (A_3^1)^{-1} & A_3^3 A_2^1 (A_3^1)^{-1} & A_3^3 \end{pmatrix}.$$

If we require that $N_a = 0$ then \mathbf{v}_a is invertible, since we have already shown that X_a are integrable. The components of the magnetic field are given by

$$\begin{aligned} B_1^1 &= \partial_2 A_3^1 - \partial_3 A_2^1 + A_2^2 A_3^3 - A_3^2 A_2^3; \\ B_2^2 &= \partial_3 A_1^2 - \partial_1 A_3^2 + A_3^3 A_1^1 - A_1^3 A_3^1; \\ B_3^3 &= \partial_1 A_2^3 - \partial_2 A_1^3 + A_1^1 A_2^2 - A_2^1 A_1^2 \end{aligned} \quad (70)$$

for the diagonal components,

$$\begin{aligned} B_2^1 &= \partial_2 A_3^2 - \partial_3 A_2^2 + A_2^3 A_3^1 - A_3^3 A_2^1; \\ B_3^2 &= \partial_3 A_1^3 - \partial_1 A_3^3 + A_3^1 A_1^2 - A_1^1 A_3^2; \\ B_1^3 &= \partial_1 A_2^1 - \partial_2 A_1^1 + A_1^2 A_2^3 - A_2^2 A_1^3 \end{aligned} \quad (71)$$

and

$$\begin{aligned} B_1^2 &= \partial_3 A_1^1 - \partial_1 A_3^1 + A_3^2 A_1^3 - A_1^2 A_3^3; \\ B_2^3 &= \partial_1 A_2^2 - \partial_2 A_1^2 + A_1^3 A_2^1 - A_2^3 A_1^1; \\ B_3^1 &= \partial_2 A_3^3 - \partial_3 A_2^3 + A_2^1 A_3^2 - A_3^1 A_2^2 \end{aligned} \quad (72)$$

for the off-diagonal components. Such configurations, where $(\det A) = 0$ with each cofactor identically vanishing, should correspond to a wide class

of integrable solutions. Note, with the cofactors vanishing, that the magnetic helicity density matrix C_{ae} reduces to

$$C_{ae} = \epsilon^{ijk} A_i^a \partial_j A_k^e. \quad (73)$$

Another way to ensure integrability of \mathbf{v}_a with nonvanishing cofactors is to require N_a to be proportional to X_a to within a numerical constant. Hence, since X_a is already integrable in accordance with the Hamiltonian flow, then N_a would as well be integrable.

3.3 Integrability in relation to Bianchi groups

In the Bianchi models one chooses a connection of the form $A^a = a_{ae} \chi^e$, where $a_{ae} = a_{ae}(t)$ is spatially constant, and all spatial dependence is absorbed into invariant one forms χ^e which satisfy the Maurer–Cartan equation

$$d\chi^e = -\frac{1}{2} C_{fg}^e \chi^f \wedge \chi^g. \quad (74)$$

In (74), C_{fg}^e are the structure constants for a group of isometries of Σ , which can be decomposed into the form

$$C_{fg}^e = n^{ed} \epsilon_{dfg} + \delta_{[f}^e a_{g]}, \quad (75)$$

where $a_g = C_{fg}^f$ is the trace of the structure constants.

The curvature two form for the connection A^a is given by

$$F^a = dA^a + \frac{1}{2} f^{abc} A^b \wedge A^c = a_{ae} d\chi^e + \frac{1}{2} f^{abc} a_{bf} a_{cg} \chi^f \wedge \chi^g. \quad (76)$$

For nondegenerate a_{ae} , (76) can be written as

$$F^a = a_{ae} d\chi^e + (\det a)(a^{-1})^{ad} \epsilon_{dfg} \chi^f \wedge \chi^g. \quad (77)$$

Using (75), (74) can be written

$$d\chi^e = -\frac{1}{2} n^{ed} \epsilon_{dfg} \chi^f \wedge \chi^g - \chi^e \wedge \chi^g a_g. \quad (78)$$

This leads to the relation

$$\epsilon_{dfg} \chi^f \wedge \chi^g = 2(n^{-1})^{de} (-d\chi^e + a_g \chi^g \wedge \chi^e). \quad (79)$$

Using the fact that the $SU(2)_-$ structure constants for the Ashtekar variables are numerically the same as the three dimensional epsilon symbol $f_{abc} = \epsilon_{abc}$, we have

$$\begin{aligned} F^a &= a_{ae} d\chi^e + (\det a)(a^{-1})^{ad} \left[-2(n^{-1})^{de} d\chi^e + 2(n^{-1})^{de} a_g \chi^g \wedge \chi^e \right] \\ &= d \left((a_{ae} - 2(\det a)(a^{-1})^{ad}(n^{-1})^{de}) \chi^e \right) + 2(\det a)(a^{-1})^{ad}(n^{-1})^{de} a_g \chi^g \wedge \chi^e. \end{aligned} \quad (80)$$

The first term on the right hand side of (80) is exact, therefore it is integrable and corresponds to a well-defined Hamiltonian flow. The second term is not manifestly exact unless $\chi^a = d\phi^a$ for some ϕ^a , which means that the Gauss' law constraint might not be solvable if it is not exact. Note for $a_g = 0$ the offending term vanishes, categorizing the Bianchi A models. This observation might perhaps shed some light on why Bianchi class A minisuperspace models are integrable while class B models, where $a_g \neq 0$ are not.

An example depicting integrability occurs when X_a and N_a are directly proportional to one another, namely $N_a = kX_a$ for some numerical constant k . We have effectively analyzed the $k = 0$ and $k = \infty$ cases, but for $0 < |k| < \infty$ we have

$$dA^a = -k f^{abc} A^b \wedge A^c. \quad (81)$$

Then $C_{ae} = \delta_{ae}(k+1)(\det A)$ which is diagonal. Hence Gauss' law constraint is exactly integrable in this case since \mathbf{v}_{ae}^{-1} exists and $\alpha_{ae} = 0$.⁸

3.4 Clebsch decomposition of the potential

As noted in [13] every three dimensional smooth vector field \vec{v} may be locally decomposed in a region free of singularities as $\vec{v} = f\vec{\nabla}g + \vec{\nabla}h$, and the local helicity of \vec{v} is given by

$$H = \vec{v} \cdot (\vec{\nabla} \times \vec{v}) = \vec{\nabla}f \cdot (\vec{\nabla}g \times \vec{\nabla}h). \quad (82)$$

According to the Frobenius theorem, if the helicity H of \vec{v} vanishes everywhere, then Clebsch functions f and g may be found such that $\vec{v} = f\vec{\nabla}g$. Such zero helicity fields are known to be completely integrable in the sense of Frobenius. We will now apply the Clebsch decomposition to the non-abelian gauge field A_i^a , thought of as a collection of three 3-vectors arranged

⁸Equation (81) is essentially the Cartan–Maurer equation where A_i^a are viewed as the components of an invariant basis of two forms. This forms essentially the basis for Bianchi models in quantum cosmology.

in the columns of a three by three matrix. Each column has a Clebsch decomposition given by

$$A^1 = f_1 dg_1 + dh_1; \quad A^2 = f_2 dg_2 + dh_2; \quad A^3 = f_3 dg_3 + dh_3, \quad (83)$$

namely $A^a = f_a dg_a + dh_a$ with no summation over a , where f_a , g_a and h_a constitute nine arbitrary smooth functions. For $f_a = g_a = 0$, and $h_a \neq 0$, we have shown already⁹ that the Gauss' law constraint is integrable. Let us now analyze the opposite extreme, namely where $h_a = 0$ with f_a and g_a not necessarily vanishing. Hence this reduces the Pfaff dimension of A_i^a from nine to six.¹⁰ For $A^a = f_a dg_a$, the magnetic field is given by

$$B^a = df_a \wedge dg_a + \frac{1}{2} f^{abc} f_b f_c dg_b \wedge dg_c. \quad (84)$$

The magnetic helicity density matrix is given, written out in terms of components, by

$$\begin{aligned} C_{ae} &= \epsilon^{ijk} f_a (\partial_i g_a) (\partial_j f_e) (\partial_k g_e) + \frac{1}{2} \epsilon^{ijk} f^{ebc} f_a f_b f_c (\partial_i g_a) (\partial_j g_b) (\partial_k g_c) \\ &= \frac{1}{2} \epsilon^{ijk} f_a (\partial_j f_e) [(\partial_i g_a) (\partial_k g_e) - (\partial_i g_e) (\partial_k g_a)] \\ &\quad + \delta_{ae} f_1 f_2 f_3 (\det(\partial g) = \delta_{ae} n + \epsilon_{aed} a_d) \end{aligned} \quad (85)$$

for some a_d and n constituting respectively, the abelian and the nonabelian contributions. One sees that for $h_a = 0$, the diagonal contributions to C_{ae} are all equal. The implication is that the matrix $\alpha_{ae} = 0$, and hence (33) reduces to the first term \mathbf{w}_{ae}^{-1} , which can in principle be carried out exactly in the event that an integrating factor is found.

This concept can be related to Papers V and VI, starting from an arbitrary potential one form $A^a = A_i^a dx^i$. While the curvature of A^a is given by

$$F^a = dA^a + \frac{1}{2} f^{abc} A^b \wedge A^c \neq 0, \quad (86)$$

which is nonvanishing for nondegenerate Ashtekar magnetic fields B_a^i , one can always find a geometric system for which A^a are the full-theory analogue of Bianchi invariant one forms. Hence there always exist structure functions $h_{bc}^a = h_{bc}^a(x)$ such that

⁹Recall the results of $A_i^a = \partial_i \phi^a$, whence one makes the identification $h_a \equiv \phi^a$. This case corresponds to a reduction of the Pfaff dimension of A_i^a from nine to three.

¹⁰This will provide as a physical interpretation the analogue of helicity for nonabelian gauge fields.

$$dA^e + \frac{1}{2}h_{bc}^e A^b \wedge A^c = 0. \quad (87)$$

Note that one may simply define

$$h_{bc}^e = (u^{-1})_i^e \mathbf{v}_{[b} \{u_c^i\}, \quad (88)$$

where $u_a^i = (A^{-1})_a^i$ for nondegenerate A_i^a . Writing the structure functions as a decomposition

$$h_{bc}^e = \epsilon_{bcf} n^{fe} + \frac{1}{2}(\delta_b^e a_c - \delta_c^e a_b) \quad (89)$$

in direct analogy to minisuperspace, one finds that the components of the magnetic helicity density are given by

$$C_{ae} d^3 x = A^a \wedge dA^e = -(\det A)(n^{ae} + \epsilon^{aed} a_d). \quad (90)$$

3.5 Sufficient condition for integrability

We will now examine some of the restrictions which can conceivably be put on the magnetic field as a consequence of integrability. Specifically, we will examine some conditions under which there exist coordinates where the vector fields have manifest integrability. The first condition is that

$$\frac{dx}{B_a^1} = \frac{dy}{B_a^2} = \frac{dz}{B_a^3}. \quad (91)$$

for $a = 1, 2, 3$. We can write (91) as an exterior differential system

$$\theta_a^3 = B_a^2 dx - B_a^1 dy; \quad \theta_a^1 = B_a^3 dy - B_a^2 dz; \quad \theta_a^2 = B_a^1 dz - B_a^3 dx. \quad (92)$$

A necessary condition that the one forms in (92) correspond to a reversible process is that the topological torsion vanish. For each a we must require that

$$\begin{aligned} \theta_a^3 \wedge d\theta_a^3 &= \left(B_a^2 \frac{\partial B_a^1}{\partial z} - B_a^1 \frac{\partial B_a^2}{\partial z} \right) dx \wedge dy \wedge dz; \\ \theta_a^1 \wedge d\theta_a^1 &= \left(B_a^2 \frac{\partial B_a^3}{\partial z} - B_a^3 \frac{\partial B_a^2}{\partial z} \right) dx \wedge dy \wedge dz; \\ \theta_a^2 \wedge d\theta_a^2 &= \left(B_a^3 \frac{\partial B_a^1}{\partial z} - B_a^1 \frac{\partial B_a^3}{\partial z} \right) dx \wedge dy \wedge dz. \end{aligned} \quad (93)$$

The requirement that each term of (93) vanish leads to the condition

$$\frac{\partial}{\partial z} \ln\left(\frac{B_a^1}{B_a^2}\right); \quad = \frac{\partial}{\partial x} \ln\left(\frac{B_a^2}{B_a^3}\right); \quad = \frac{\partial}{\partial z} \ln\left(\frac{B_a^3}{B_a^1}\right) = 0. \quad (94)$$

Equation (94) leads to the conditions

$$\frac{B_a^1}{B_a^2} = c_a(x, y); \quad \frac{B_a^2}{B_a^3} = a_a(x, y); \quad \frac{B_a^3}{B_a^1} = b_a(z, x) \quad (95)$$

for each a for some functions (a_a, b_a, c_a) . Additionally, we have that

$$a_a b_a c_a = 1 \quad \forall a. \quad (96)$$

We can use (95) to solve for all off-diagonal elements in terms of the diagonal elements, obtaining

$$B_a^i = \begin{pmatrix} 1 & c_2(x, y) & b_3^{-1}(x, y) \\ c_1^{-1}(x, y) & 1 & a_3(y, z) \\ b_1(z, x) & a_2^{-1}(y, z) & 1 \end{pmatrix} \begin{pmatrix} B_1^1(x, y, z) & 0 & 0 \\ 0 & B_2^2(x, y, z) & 0 \\ 0 & 0 & B_3^3(x, y, z) \end{pmatrix}.$$

We can use (96) to eliminate c_1 and c_2 , making the re-definition $a_2^{-1} \rightarrow a_2$ and $b_3^{-1} \rightarrow b_3$, obtaining

$$B_a^i = \begin{pmatrix} 1 & a_2(x, y)b_2(z, x) & b_3(x, y) \\ a_1(y, z)b_1^{-1}(z, x) & 1 & a_3(y, z) \\ b_1(z, x) & a_2(y, z) & 1 \end{pmatrix} \begin{pmatrix} B_1^1(x, y, z) & 0 & 0 \\ 0 & B_2^2(x, y, z) & 0 \\ 0 & 0 & B_3^3(x, y, z) \end{pmatrix}.$$

We have obtained a decomposition of the magnetic field into a part which is restricted by integrability requirements, and a part which is completely unrestricted with three degrees of freedom per point. One interesting question is whether this restriction makes sense when B_a^i plays the dual role of a configuration space dynamical variable as in GR. According to the results of Paper VI there appears to be no obstruction to any arbitrary $\text{GL}(3)$ -valued matrix B_a^i playing the role of a magnetic field. This is because one can always find a connection A_i^a such that B_a^i is the magnetic field, and then the helicity density C_{ae} is a derived quantity. So certainly, one can in principle find A_i^a that yields B_a^i of the above form.

4 A few examples

The Hamiltonian and diffeomorphism constraints can be solved algebraically without reference to any coordinate system, and reduce Ψ_{ae} by four degrees of freedom upon solution. The Gauss' law contains spatial gradients ∂_i intermingled with B_a^i in the form of vector fields \mathbf{v}_a . The solution of this constraint requires inversion of the differential operators, which requires the specification of a configuration of A_i^a under which the inversion will be performed. More fundamentally, one must specify which congruence $\vec{\gamma}$ of integral curves filling 3-space Σ one wishes to consider. Note that $\vec{\gamma}$ and the tangent vector fields \mathbf{v}_a are coordinate-independent. The specification of a configuration A_i^a then amounts to a choice of coordinates x^i as follows. Given $\vec{\gamma}$ and X^i , then $\mathbf{v}_a\{x^i\} = B_a^i$ determines the components of \mathbf{v}_a in a particular coordinate system. The requirement that B_a^i be a magnetic field then fixes A_i^a as one member of an equivalence class of potentials defined by B_a^i such that the Bianchi identity is satisfied. Hence A_i^a in a sense is intertwined with the choice of coordinate system. To solve the Gauss' law constraint then amounts to the imposition of integrability on \mathbf{v}_a , derived from $\vec{\gamma}$, which are regarded as the fundamental structure. The CDJ matrix Ψ_{ae} suffers a reduction in three D.O.F. upon implementation of G_a , leaving only three D.O.F. which can be considered physical in the sense that they are independent of the choice of coordinates.

The Gauss' law constraint can be written as

$$\mathbf{w}_a^{fg}\{\Psi_{fg}\} = (\delta^{af}(\mathbf{v}_g - f_{gbe}C_{be}) + f_{abf}C_{bg})\Psi_{fg}, \quad (97)$$

where the vector fields \mathbf{v}_g satisfy (7). The homogeneous reduction can best be understood by writing the Ashtekar magnetic field in the form

$$B_a^i = \epsilon^{ijk}\partial_j A_k^a + (\det A)(A^{-1})_a^i, \quad (98)$$

depicting an 'abelian' part involving spatial gradients, plus a nonabelian correction. This relation follows from the fact that the structure constants $f_{abc} = \epsilon_{abc}$ for $SU(2)_-$ are numerically the same as the epsilon symbol ϵ_{ijk} . Likewise, the following suggestive form for the magnetic helicity density matrix

$$C_{be} = \epsilon^{ijk}A_i^b\partial_j A_k^e + \delta^{be}(\det A) \quad (99)$$

can be written. Inserting (98) and (99) into (97), we obtain

$$\mathbf{w}_a^{fg} = (\det A)(\delta^{af}(A^{-1})_g^i\partial_i + f_{agf}) + r_a^{fg} \quad (100)$$

where the remainder r_a^{fg} , which contains the gradient terms, is given by

$$r_a^{fg} = \epsilon^{ijk} \left(\delta^{af} (\partial_j A_k^g) \partial_i + (\delta_{af} f_{bge} + \delta_{eg} f_{abf}) A_i^b \partial_j A_k^e \right). \quad (101)$$

4.1 Spatially homogeneous sector of connection superspace

We now reduce the constraints to the spatially homogeneous sector of connection superspace, by choosing a configuration for which A_i^a are spatially constant.¹¹ Note from (99) that the only nonvanishing parts of C_{be} are $C_{be} = \delta_{be}(\det A)$. Setting the spatial gradients to zero, we have $r_a^{fg} = 0$ which leads to

$$\mathbf{w}_a^{fg} = (\det A) (\delta^{af} (A^{-1})_g^i \partial_i + f_{agf}). \quad (102)$$

Since Ψ_{fg} is symmetric, then the Gauss' law constraint is given by

$$\mathbf{w}_a^{fg} \{\Psi_{fg}\} = (\det A) (A^{-1})_g^i \partial_i \{\Psi_{ag}\} = 0. \quad (103)$$

For spatially constant B_a^i , we have the relation $B_a^i = (\det A) (A^{-1})_a^i$. We may now rescale the constraint by a factor of $\det A \neq 0$, since it is now invariant under rescaling. Making the identification $\mathbf{e}_a = (A^{-1})_a^i \partial_i$ for spatially constant A_i^a , we have that $[\mathbf{e}_a, \mathbf{e}_b] = 0$, where

$$\mathbf{e}_a \equiv \frac{\partial}{\partial t^a}; \quad \mathbf{e}_a^{-1} \equiv \int dt^a. \quad (104)$$

Hence the structure functions h_{ab}^e in (8) all vanish. The following relations additionally ensue¹²

$$[\mathbf{e}_a, \mathbf{e}_b] = [\mathbf{e}_a^{-1}, \mathbf{e}_b^{-1}] = [\mathbf{e}_a^{-1}, \mathbf{e}_b] = 0. \quad (105)$$

Since the matrix elements of the Gauss' law kinetic operator commute, then its inversion is analogous to the inversion of a matrix of c-numbers.

The integral curves of \mathbf{e}_a are given by

¹¹Minisuperspace, by our definition, means that spatial gradients of all quantities are vanishing. By this definition, the spatially homogeneous sector of A_i^a does not necessarily imply minisuperspace, since as we will see that Ψ_{ae} needn't be spatially constant. In the spatially homogeneous sector the identity $\det B = (\det A)^2$ follows from (98)

¹²We set the constants of antidifferentiation to zero, so that \mathbf{e}_a^{-1} is the unique inverse of \mathbf{e}_a . We will adopt the same convention in general when solving the constraint.

$$\frac{dx^i}{dt^a} = (\det A)(A^{-1})_a^i \longrightarrow x^i = (\det A)(A^{-1})_a^i t^a, \quad (106)$$

which are straight lines in 3-space Σ . Let us define a ‘densitized’ version $\tilde{t}^a = t^a(\det A)$, of the coordinates. Then the following transformation defines a nonorthogonal coordinate system

$$\begin{aligned} \tilde{t}^1 &= A_1^1 x + A_2^1 y + A_3^1 z; \\ \tilde{t}^2 &= A_1^2 x + A_2^2 y + A_3^2 z; \\ \tilde{t}^3 &= A_1^3 x + A_2^3 y + A_3^3 z, \end{aligned} \quad (107)$$

which allows the CDJ matrix $\Psi_{fg} = \Psi_{fg}(\tilde{t}^1, \tilde{t}^2, \tilde{t}^3)$ to be expressed directly in this coordinate system without reference to (x, y, z) .

The matrix form of the Gauss’ law constraint (103) is given by

$$\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & \mathbf{e}_1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} + \begin{pmatrix} \mathbf{e}_1 & 0 & 0 \\ 0 & \mathbf{e}_2 & 0 \\ 0 & 0 & \mathbf{e}_3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0.$$

The solution is given by

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = - \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & \mathbf{e}_1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{e}_1 & 0 & 0 \\ 0 & \mathbf{e}_2 & 0 \\ 0 & 0 & \mathbf{e}_3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

In the spatially homogeneous sector of B_a^i the matrices of differential operators behave the same way as c numbers, when the boundary data for integration is set to zero. This leads to the relation

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (\mathbf{e}_1)^2(\mathbf{e}_2\mathbf{e}_3)^{-1} & -\mathbf{e}_2\mathbf{e}_3^{-1} & -\mathbf{e}_3\mathbf{e}_2^{-1} \\ -\mathbf{e}_1\mathbf{e}_3^{-1} & (\mathbf{e}_2)^2(\mathbf{e}_3\mathbf{e}_1)^{-1} & -\mathbf{e}_3\mathbf{e}_1^{-1} \\ -\mathbf{e}_1\mathbf{e}_2^{-1} & -\mathbf{e}_2\mathbf{e}_1^{-1} & (\mathbf{e}_3)^2(\mathbf{e}_1\mathbf{e}_2)^{-1} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

The Gauss’ law constraint must be solved in conjunction with the Hamiltonian constraint, given by

$$\frac{1}{2}Var(\Psi_{ae}) + \Lambda\det(\Psi_{ae}) = 0. \quad (108)$$

Due to the invariance of the quadratic part of (108) under rescaling, we will consider a special case. Define

$$\Psi_a = m_a e^{\vec{k} \cdot \vec{r}}; \quad \varphi_a = n_a e^{\vec{k} \cdot \vec{r}}, \quad (109)$$

where m_a and n_a are spatially constant.¹³ The the eigenvalue of the action of e_a and its inverse on the exponential in (109) by

$$e_a \{e^{\vec{k} \cdot \vec{r}}\} = (A^{-1})_a^j \frac{\partial}{\partial x^j} e^{\vec{k} \cdot \vec{r}} = e_a e^{\vec{k} \cdot \vec{r}}, \quad (110)$$

where we have defined $e_a = (A^{-1})_a^j k_j$. Similarly,

$$e_a^{-1} \{e^{\vec{k} \cdot \vec{r}}\} = A_j^a \int dx^j \{e^{\vec{k} \cdot \vec{r}}\} = e_a^{-1} \{e^{\vec{k} \cdot \vec{r}}\}, \quad (111)$$

where we have defined $e_a^{-1} = \sum_j A_j^a / k_j$. Note that only the projection $(A^{-1})_a^i$ onto the wave vector k_i enters into the Gauss' law constraint.

The Gauss' law constraint then reduces to

$$\begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -e_1 e_2^{-1} & -e_2 e_1^{-1} & -e_3^2 (e_1 e_2)^{-1} \\ -e_1^2 (e_2 e_3)^{-1} & -e_2 e_3^{-1} & -e_3 e_2^{-1} \\ -e_1 e_3^{-1} & -e_2^2 (e_3 e_1)^{-1} & -e_3 e_1^{-1} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

It is convenient for calculational purposes to define $e_{ae} \equiv e_a (e_b)^{-1}$, then the map from the anisotropy to the shear elements is given by

$$\begin{aligned} \Psi_3 &= \frac{1}{2} (-e_{12} \varphi_1 - e_{21} \varphi_2 + e_{31} e_{32} \varphi_3); \\ \Psi_1 &= \frac{1}{2} (e_{12} e_{13} \varphi_1 - e_{23} \varphi_2 - e_{32} \varphi_3); \\ \Psi_2 &= \frac{1}{2} (-e_{13} \varphi_1 + e_{23} e_{21} \varphi_2 - e_{31} \varphi_3). \end{aligned} \quad (112)$$

The Hamiltonian constraint is given by

$$H = \frac{1}{2} Var \Psi + \Lambda \det \Psi = 0. \quad (113)$$

The ingredients for (113) in terms of the CDJ matrix elements are given, starting with the variance, by

¹³Note that this an example of a configuration where A_i^a are spatially homogeneous, but Ψ_{ae} are not.

$$\begin{aligned} \frac{1}{2}Var\Psi &= \varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_3\varphi_1 - (\Psi_1^2 + \Psi_2^2 + \Psi_3^2); \\ &= \frac{|e|^2}{4} \left[\left(\frac{e_1}{e_2e_3} \right)^2 \varphi_1^2 + \left(\frac{e_2}{e_3e_1} \right)^2 \varphi_2^2 + \left(\frac{e_3}{e_1e_2} \right)^2 \varphi_3^2 - 2 \left(\frac{\varphi_1\varphi_2}{e_3^2} + \frac{\varphi_2\varphi_3}{e_1^2} + \frac{\varphi_3\varphi_1}{e_2^2} \right) \right] \end{aligned} \quad (114)$$

where we have defined $|e|^2 = (e_1)^2 + (e_2)^2 + (e_3)^2$. The determinant is given by

$$\det\Psi = \varphi_1\varphi_2\varphi_3 + 2\Psi_1\Psi_2\Psi_3 - (\varphi_1\Psi_1^2 + \varphi_2\Psi_2^2 + \varphi_3\Psi_3^2) \quad (115)$$

We will show the intermediate stages of the calculation of (112). First we have

$$\begin{aligned} 2\Psi_1\Psi_2\Psi_3 + \varphi_1\varphi_2\varphi_3 &= \frac{1}{4}(A + \varphi_1\varphi_2\varphi_3); \\ \varphi_1\Psi_1^2 + \varphi_2\Psi_2^2 + \varphi_3\Psi_3^2 &= \frac{1}{4}(A + 6\varphi_1\varphi_2\varphi_3), \end{aligned} \quad (116)$$

where A is given by

$$\begin{aligned} A &= (e_{12}e_{13})^2\varphi_1^3 - (e_{13})^2\varphi_2\varphi_1\varphi_2 - (e_{12})^2\varphi_1\varphi_1\varphi_3 \\ &\quad + (e_{23}e_{21})^2\varphi_2^2 - (e_{23})\varphi_2\varphi_2\varphi_1 - (e_{23})^2\varphi_2\varphi_2\varphi_1 \\ &\quad + (e_{31}e_{32})^2\varphi_3^2 - (e_{32})^2\varphi_3\varphi_3\varphi_1 - (e_{31})^2\varphi_3\varphi_3\varphi_2. \end{aligned} \quad (117)$$

The main observation is that the difference between the first and the second line of (115), which is $\det\Psi$, vanishes. By virtue of the choice of configuration the CDJ matrix is degenerate, and is insensitive to the cosmological constant. The Hamiltonian constraint (113) then reduces to the variance, which is the condition that (114) be zero. This is simply a quadratic equation in φ_3 with roots

$$\varphi_3 = \left[\left(\frac{e_1}{e_3} \right) \sqrt{\varphi_1} \pm \left(\frac{e_2}{e_3} \right) \sqrt{\varphi_2} \right]^2. \quad (118)$$

Substitution of (118) into (112) yields the shear CDJ matrix elements

$$\begin{aligned} \Psi_3 &= \frac{1}{2} \left(\left(\frac{e_3 - e_1}{e_2} \right) \varphi_1 + \left(\frac{e_3 - e_2}{e_1} \right) \varphi_2 \pm 2\sqrt{\varphi_1\varphi_2} \right); \\ \Psi_1 &= \frac{1}{2} \left(-\frac{e_1}{e_2} \left(1 - \frac{e_1}{e_3} \right) \varphi_1 - \left(1 + \frac{e_2}{e_3} \right) \varphi_2 \mp \frac{e_1}{e_3} \sqrt{\varphi_1\varphi_2} \right); \\ \Psi_2 &= \frac{1}{2} \left(-\left(1 + \frac{e_1}{e_3} \right) \varphi_1 - \frac{e_2}{e_1} \left(1 - \frac{e_2}{e_3} \right) \varphi_2 \mp \frac{e_2}{e_3} \sqrt{\varphi_1\varphi_2} \right). \end{aligned} \quad (119)$$

Since the CDJ matrix is degenerate, then the 3-metric h_{ij} cannot be constructed. Nevertheless, the solution for Ψ_{ae} is well-defined and provided in terms of φ_1 and φ_2 , regarded as the two physical degrees of freedom per point. The information regarding the connection is encoded in the coefficients e_a , and is coordinate-independent.

4.2 Passive diffeomorphism configurations

The previous example is a special case of the following situation where the connection A_i^a is not spatially homogeneous, but corresponds to the Jacobian matrix of a transformation between two coordinate systems. For each point P in the 3-dimensional spatial manifold Σ , there exists a homeomorphism from Σ to R^3 which expresses the coordinates of P in a local chart. The atlas of Σ consists of the set of all local charts of Σ . We will associate each chart to the Ashtekar configuration space variables as follows. Define an invertible transformation from a chart depicting local coordinates x^i into a chart depicting local coordinates ϕ^e , where

$$\phi^1 = \phi^1(x, y, z); \quad \phi^2 = \phi^2(x, y, z); \quad \phi^3 = \phi^3(x, y, z). \quad (120)$$

The transition functions define a Jacobian matrix J , given by

$$J_i^a = \left(\frac{\partial \phi^a}{\partial x^i} \right), \quad (121)$$

and for J invertible, the map $\vec{x} \rightarrow \vec{\phi}(\vec{x})$ constitutes a homeomorphism. Now choose the connection to be the Jacobian matrix

$$A_i^a = \left(\frac{\partial \phi^a}{\partial x^i} \right). \quad (122)$$

The choice (122) can be thought of as a triple of flat abelian gauge fields. However, the magnetic field is nonvanishing since it is a nonabelian gauge potential. The Ashtekar magnetic field is given by

$$B_a^i = \epsilon^{ijk} \partial_j \left(\frac{\partial \phi^a}{\partial x^k} \right) + (\det J) \left(\frac{\partial x^i}{\partial \phi^a} \right) = (\det J) \left(\frac{\partial x^i}{\partial \phi^a} \right), \quad (123)$$

where the abelian part vanishes and we are left with the nonabelian part. The helicity density matrix is given by

$$C_{ae} = J \left(\frac{\partial \phi^a}{\partial x^i} \right) \left(\frac{\partial x^i}{\partial \phi^e} \right) = \delta_{ae} J. \quad (124)$$

Since C_{ae} is isotropic, this configuration is definitely integrable. The vector fields are given by

$$\mathbf{v}_a = J \left(\frac{\partial x^i}{\partial \phi^a} \right) \frac{\partial}{\partial x^i} = \frac{\partial}{\partial \phi^a}, \quad (125)$$

which are exactly invertible since they are variations with respect to local coordinates.

To show that the structure functions needn't be numerical constants. let us compute the algebra of vector fields

$$[\mathbf{v}_a, \mathbf{v}_b] = h_{ab}^f \mathbf{v}_f. \quad (126)$$

Expanding this we obtain

$$\begin{aligned} & \left[\left(\frac{\partial x^i}{\partial \phi^a} \right) \frac{\partial}{\partial x^i}, \left(\frac{\partial x^j}{\partial \phi^b} \right) \frac{\partial}{\partial x^j} \right] \\ &= \left(\frac{\partial \phi^e}{\partial x^i} \right) \left(\frac{\partial \phi^f}{\partial x^j} \right) \left[\frac{\partial x^i}{\partial \phi^a} \frac{\partial^2 x^j}{\partial t^e \partial \phi^b} - \frac{\partial x^i}{\partial \phi^b} \frac{\partial^2 x^j}{\partial \phi^e \partial t^a} \right] \left(\frac{\partial \phi^f}{\partial x^j} \right) \frac{\partial}{\partial \phi^f}. \end{aligned} \quad (127)$$

From (127) we read off the structure functions as

$$h_{ab}^f dx^a \wedge dx^b = \left(\frac{\partial \phi^e}{\partial x^i} \right) \left(\frac{\partial \phi^f}{\partial x^j} \right) \{ dx^i, d(\partial x^j / \partial \phi^e) \}, \quad (128)$$

which take on the interpretation of the Poisson bracket between the coordinate and its velocities with respect to the new coordinate system.

The Gauss' law constraint for this class of configurations is given in matrix form by

$$\begin{pmatrix} \frac{\partial}{\partial \phi^2} & 0 & \frac{\partial}{\partial \phi^3} \\ \frac{\partial}{\partial \phi^1} & \frac{\partial}{\partial \phi^3} & 0 \\ 0 & \frac{\partial}{\partial \phi^2} & \frac{\partial}{\partial \phi^1} \end{pmatrix} \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial \phi^1} & 0 & 0 \\ 0 & \frac{\partial}{\partial \phi^2} & 0 \\ 0 & 0 & \frac{\partial}{\partial \phi^3} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0.$$

Note that the magnetic helicity density contribution is zero, hence the integration of this system should be relatively simple. To solve this it is expedient to map $\vec{\varphi}$ into the corresponding coordinate system

$$\varphi_f(x, y, z) \rightarrow \varphi_f(\phi^1, \phi^2, \phi^3), \quad (129)$$

since the coordinates \vec{t} are globally well-defined within the domain of definition of the passive diffeomorphism. The solution is then given by

$$\begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = - \begin{pmatrix} \frac{\partial}{\partial \phi^2} & 0 & \frac{\partial}{\partial \phi^3} \\ \frac{\partial}{\partial \phi^1} & \frac{\partial}{\partial \phi^3} & 0 \\ 0 & \frac{\partial}{\partial \phi^2} & \frac{\partial}{\partial \phi^1} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial \phi^1} & 0 & 0 \\ 0 & \frac{\partial}{\partial \phi^2} & 0 \\ 0 & 0 & \frac{\partial}{\partial \phi^3} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

The cyclic operators are given by

$$\begin{aligned}\pi_2 &= \partial_2^{-1} \partial_3 \partial_1^{-1} \partial_2 \partial_3^{-1} \partial_1 \equiv 1; \\ \pi_3 &= \partial_3^{-1} \partial_1 \partial_2^{-1} \partial_3 \partial_1^{-1} \partial_2 \equiv 1; \\ \pi_1 &= \partial_1^{-1} \partial_2 \partial_3^{-1} \partial_1 \partial_2^{-1} \partial_3 \equiv 1,\end{aligned}\tag{130}$$

which are the identity since the vector fields and their inverses take on the clear interpretation of differentiation and antidifferentiation with respect to the chosen coordinate system. The intermediate stages of inversion of the shear subspace kinetic operator is given by

$$\begin{aligned}u^{-1} &= \frac{1}{2} \begin{pmatrix} \partial_2^{-1} & 0 & 0 \\ 0 & \partial_3^{-1} & 0 \\ 0 & 0 & \partial_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & \partial_3 \partial_1^{-1} \partial_2 \partial_3^{-1} & -\partial_3 \partial_1^{-1} \\ -\partial_1 \partial_2^{-1} & 1 & \partial_1 \partial_2^{-1} \partial_3 \partial_1^{-1} \\ \partial_2 \partial_3^{-1} \partial_1 \partial_2^{-1} & -\partial_2 \partial_3^{-1} & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \partial_2^{-1} & 0 & 0 \\ 0 & \partial_3^{-1} & 0 \\ 0 & 0 & \partial_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & \partial_1^{-1} \partial_2 & -\partial_3 \partial_1^{-1} \\ -\partial_1 \partial_2^{-1} & 1 & \partial_2^{-1} \partial_3 \\ \partial_3^{-1} \partial_1 & -\partial_2 \partial_3^{-1} & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \partial_2^{-1} & \partial_1^{-1} & -\partial_2^{-1} \partial_3 \partial_1^{-1} \\ -\partial_3^{-1} \partial_1 \partial_2^{-1} & \partial_3^{-1} & \partial_2^{-1} \\ \partial_3^{-1} & -\partial_1^{-1} \partial_2 \partial_3^{-1} & \partial_1^{-1} \end{pmatrix}.\end{aligned}$$

From this we can construct the Gauss' law propagator

$$\hat{J}_f^g = \frac{1}{2} \begin{pmatrix} -\partial_1 \partial_2^{-1} & -\partial_1^{-1} \partial_2 & (\partial_3)^2 (\partial_1 \partial_2)^{-1} \\ (\partial_1)^2 (\partial_2 \partial_3)^{-1} & -\partial_2 \partial_3^{-1} & -\partial_3 \partial_2^{-1} \\ -\partial_1 \partial_3^{-1} & (\partial_2)^2 (\partial_3 \partial_1)^{-1} & -\partial_3 \partial_1^{-1} \end{pmatrix}.$$

and the general solution to the Gauss' law constraint is given by¹⁴

$$\Psi_{ae} = T_{ae}^g \varphi_g \equiv T_{ae}[\vec{\varphi}; \vec{\gamma}]; \quad T_{ae}^g = e_{ae}^g + E_{ae}^f \hat{J}_f^g[\vec{\gamma}].\tag{131}$$

The Hamiltonian constraint can then be written as a condition on the reduced space of diagonal CDJ matrix elements as

$$\hat{Q}^{f'g'} \varphi_{f'} \varphi_{g'} + \hat{Q}^{f'g'h'} \varphi_{f'} \varphi_{g'} \varphi_{h'} = 0,\tag{132}$$

¹⁴Equation (131) and what follows can be taken as the general notation for all cases, not limited to the one being considered in this example.

where we have defined the following operators from the invariant tensors of the unconstrained space Ψ_{ae} ,

$$\hat{Q}^{f'g'} = \frac{1}{2}(\delta^{ae}\delta^{bf} - \delta^{af}\delta^{be})T_{ae}^{f'}T_{bf}^{g'} \quad (133)$$

from the variance, and

$$\hat{Q}^{f'g'h'} = \frac{\Lambda}{6}\epsilon^{abc}\epsilon^{efg}T_{ae}^{f'}T_{bf}^{g'}T_{cg}^{h'} \quad (134)$$

from the determinant. Equations (133) and (134) can be seen as embedding maps from the physical degrees of freedom, which are independent of coordinates, into the bigger space which takes into account the kinematic effects of the Gauss' law constraint. The inverse map constitutes the projection to these physical degrees of freedom, of which two per point are freely specifiable. This can be realized when one solved (131) to yield

$$\varphi_3 = \varphi_3(\varphi_1, \varphi_2; \vec{\gamma}). \quad (135)$$

The notation in (135) indicates that each solution can be expressed in terms of two degrees of freedom per point φ_1 and φ_2 , in conjunction with the equivalence class of integral curves $\vec{\gamma}$ which completely define the Gauss' law constraint. The spatial 3-metric can be reconstructed from the solution to the initial value constraints via the relation

$$h_{ij} = \eta^{ae}(B^{-1})_i^a(B^{-1})_j^e(\det B), \quad (136)$$

where we have defined the internal metric η^{ae} by

$$\eta^{ae} = (\det \Psi)(\Psi^{-1}\Psi^{-1})^{ae} = (\det T[\vec{\phi}])(T^{-1}[\vec{\phi}]T^{-1}[\vec{\phi}])^{-1} = \eta^{ae}[\varphi_1, \varphi_2; \vec{\gamma}] \quad (137)$$

The metric is given by

$$h_{ij} = \eta^{ae} \frac{\partial \phi^a}{\partial x^i} \frac{\partial \phi^e}{\partial x^j} = h_{ij}[\varphi_1, \varphi_2; \vec{\gamma}]. \quad (138)$$

One could present the argument that all magnetic fields B_a^i within the class of passive diffeomorphism configurations fall within the same equivalence class of the constant field $B_a^i = \delta_{ia}$, since they are obtainable by coordinate transformation. However, the metric still has a complicated nonlinear

dependence on the particular B_a^i within this class, owing to the nonlinear Hamiltonian constraint. Equation (132) can be solved formally using Feynman–diagrammatic methods, which we will not display here.

The previous example treated is a special case of the passive diffeomorphism configuration. Corresponding to each general coordinate transformation is a configuration solving the Gauss’ law constraint as indicated. To complete the solution we incorporate the Hamiltonian constraint.

4.3 t'Hooft Polyakov monopole

We will now consider configurations A_i^a which do not correspond to a well-defined map between two charts, for which the Gauss' law constraint is still integrable. Let us examine the case of a nonabelian magnetic monopole

$$A_i^a = \epsilon_{aij} \frac{g x^j}{r^2}, \quad (139)$$

of monopole strength g , where in this example and the remaining examples to follow, $r^2 = x^2 + y^2 + z^2$.¹⁵ The magnetic field is given by

$$B_a^i = -\left(\frac{2g}{r^4}\right)x^a x^i + \frac{g^2}{r^4}x^a x^i = \frac{g(g-2)}{r^4}x^a x^i, \quad (140)$$

and the helicity density matrix C_{ae} is given by

$$C_{ae} = \frac{g^2(g-2)}{r^6}(\epsilon_{aij}x^j x^e x^i) = 0, \quad (141)$$

which vanishes due to antisymmetry of ϵ_{ijk} . Therefore we should expect this configuration to be integrable.

The magnetic field can be decomposed into a product $B_a^i = R_j^i(\delta_{ja}B_a^a)$, where

$$R_j^i = \begin{pmatrix} 1 & \frac{x}{y} & \frac{x}{z} \\ \frac{y}{x} & 1 & \frac{y}{z} \\ \frac{z}{x} & \frac{z}{y} & 1 \end{pmatrix}; \quad \delta_{ja}B_a^a = \frac{g(g-2)}{r^4} \begin{pmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{pmatrix}.$$

This suggests that $B_a^a \propto (x^a)^2$ is the part of B_a^i not constrained by coordinate system. The part intertwined with the coordinate system is R_j^i . Note that $\det(R_j^i) = 0$, causing B_a^i to be degenerate. Nevertheless, we can still solve the constraints.

Note from (140) that $N_a \propto X_a$, yielding a vector field

$$\mathbf{v}_a = X_a + N_a = \frac{g(g-2)}{r^4}x^a x^i \frac{\partial}{\partial x^i}. \quad (142)$$

Since X_a is integrable, and $N_a \propto X_a$, it follows that N_a and therefore \mathbf{v}_a is integrable. Indeed, defining $v_a = x^a x^i \partial_i$, one sees that $[v_a, v_b] = 0$ with vanishing structure functions h_{ab}^e . The integral curves of $x^i \partial_i \equiv \frac{d}{dt}$, correspond to a flow

¹⁵This can also be seen as an instanton of zero size.

$$\phi_t(x, y, z) = (x_0 e^t, y_0 e^t, z_0 e^t). \quad (143)$$

Cancelling off a factor of r^{-4} from (142) and using $C_{ae} = 0$, the Gauss law constraint for $g \neq 2$ reduces to the form

$$\begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix} \begin{pmatrix} d\Psi_1/dt \\ d\Psi_2/dt \\ d\Psi_3/dt \end{pmatrix} = - \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} d\varphi_1/dt \\ d\varphi_2/dt \\ d\varphi_3/dt \end{pmatrix}.$$

We have defined a parameter t to denote location along the integral curves of the vector field $x^i \partial_i$, which acts as the identity operator on coordinates $\frac{dx^i}{dt} = x^i$. Using the relation

$$x^i \frac{d\Psi}{dt} = \frac{d}{dt}(x^i \Psi) - \left(\frac{dx^i}{dt}\right) \Psi = \left(\frac{d}{dt} - 1\right)(x^i \Psi), \quad (144)$$

we can reduce the Gauss' law constraint to the form

$$\left(\frac{d}{dt} - 1\right) \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = - \left(\frac{d}{dt} - 1\right) \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

Eliminating the differential operator $\frac{d}{dt} - 1$ from both sides, we can algebraically solve for $\Psi_a = \Psi_a[\vec{\varphi}]$

$$\begin{aligned} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} &= - \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}^{-1} \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \frac{x^2}{yz} & -\frac{y}{z} & -\frac{z}{y} \\ -\frac{x}{z} & \frac{y^2}{zx} & -\frac{z}{x} \\ -\frac{x}{y} & -\frac{y}{x} & \frac{z^2}{xy} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}. \end{aligned}$$

Expanding this out we have

$$\begin{aligned} 2\Psi_1 &= \left(\frac{x^2}{yz}\right)\varphi_1 - \left(\frac{y}{z}\right)\varphi_2 - \left(\frac{z}{y}\right)\varphi_3; \\ 2\Psi_2 &= -\left(\frac{x}{z}\right)\varphi_1 + \left(\frac{y^2}{zx}\right)\varphi_2 - \left(\frac{z}{x}\right)\varphi_3; \\ 2\Psi_3 &= -\left(\frac{x}{y}\right)\varphi_1 - \left(\frac{y}{x}\right)\varphi_2 + \left(\frac{z^2}{xy}\right)\varphi_3. \end{aligned} \quad (145)$$

In conjunction with solving the Gauss' law constraint, one must also solve the Hamiltonian constraint in order to obtain a full solution to the initial value constraints. The Hamiltonian constraint is given by

$$\begin{aligned} Var\Psi + \Lambda det\Psi &= 2(\varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_3\varphi_1 - \Psi_1^2 - \Psi_2^2 - \Psi_3^2) \\ &+ \Lambda(\varphi_1\varphi_2\varphi_3 + 2\Psi_1\Psi_2\Psi_3 - \varphi_1\Psi_1^2 - \varphi_2\Psi_2^2 - \varphi_3\Psi_3^2) = 0. \end{aligned} \quad (146)$$

Since the relation between $\vec{\Psi}$ and $\vec{\varphi}$ is algebraic, then the Hamiltonian constraint can be solved exactly in closed form. We will do this, by substituting (145) into (146). The intermediate stages of the calculation are as follows. For the shear contribution to the variance we have

$$\begin{aligned} \Psi_1^2 + \Psi_2^2 + \Psi_3^2 &= \frac{1}{4} \left[\left(\frac{x^2 r^2}{y^2 z^2} \right) \varphi_1^2 + \left(\frac{y^2 r^2}{z^2 x^2} \right) \varphi_2^2 + \left(\frac{z^2 r^2}{x^2 y^2} \right) \varphi_3^2 \right. \\ &+ 2 \left(\left(\frac{z^2 - x^2 - y^2}{z^2} \right) \varphi_1 \varphi_2 + \left(\frac{x^2 - y^2 - z^2}{x^2} \right) \varphi_2 \varphi_3 + \left(\frac{y^2 - z^2 - x^2}{y^2} \right) \varphi_3 \varphi_1 \right] \end{aligned} \quad (147)$$

The shear contribution to the determinant is given, after a long algebraic calculation, by

$$\begin{aligned} \Psi_1\Psi_2\Psi_3 &= \frac{1}{8} \left[\left(\frac{x^4}{y^2 z^2} \right) \varphi_1^3 + \left(\frac{y^4}{z^2 x^2} \right) \varphi_2^3 + \left(\frac{z^4}{x^2 y^2} \right) \varphi_3^3 + 2\varphi_1\varphi_2\varphi_3 \right. \\ &- \left(\frac{x^2}{z^2} \right) \varphi_1^2 \varphi_2 - \left(\frac{y^2}{z^2} \right) \varphi_1 \varphi_2^2 - \left(\frac{y^2}{x^2} \right) \varphi_2^2 \varphi_3 - \left(\frac{z^2}{x^2} \right) \varphi_2 \varphi_3^2 - \left(\frac{z^2}{y^2} \right) \varphi_3^2 \varphi_1 - \left(\frac{x^2}{y^2} \right) \varphi_3 \varphi_1^2 \left. \right] \end{aligned} \quad (148)$$

The other contribution to the determinant besides $\varphi_1\varphi_2\varphi_3$ is given by

$$\begin{aligned} &\varphi_1\Psi_1^2 + \varphi_2\Psi_2^2 + \varphi_3\Psi_3^2 \\ &= \frac{1}{4} \left[\left(\frac{x^4}{y^2 z^2} \right) \varphi_1^3 + \left(\frac{y^4}{z^2 x^2} \right) \varphi_2^3 + \left(\frac{z^4}{x^2 y^2} \right) \varphi_3^3 + 2\varphi_1\varphi_2\varphi_3 \right. \\ &\quad - \left(\frac{x^2}{z^2} \right) \varphi_1^2 \varphi_2 - \left(\frac{y^2}{z^2} \right) \varphi_1 \varphi_2^2 - \left(\frac{y^2}{x^2} \right) \varphi_2^2 \varphi_3 \\ &\quad \left. - \left(\frac{z^2}{x^2} \right) \varphi_2 \varphi_3^2 - \left(\frac{z^2}{y^2} \right) \varphi_3^2 \varphi_1 - \left(\frac{x^2}{y^2} \right) \varphi_3 \varphi_1^2 + 6\varphi_1\varphi_2\varphi_3 \right]. \end{aligned} \quad (149)$$

We have shown the intermediate stages of the calculation to point out a remarkable effect which happens for this configuration. Upon insertion of the results of (147), (148) and (149) into (146), the coefficient of Λ vanishes due to a precise cancellation and the Hamiltonian constraint reduces from a cubic to the following quadratic equation

$$\varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_3\varphi_1 - (\Psi_1)^2 - (\Psi_2)^2 - (\Psi_3)^2 = 0. \quad (150)$$

The result is that the monopole configuration is insensitive to the presence of a cosmological constant. Putting (145) into the right hand side of (150) we obtain the following quadratic equation

$$\begin{aligned} & \left(\frac{x^2}{y^2z^2}\right)\varphi_1^2 + \left(\frac{y^2}{z^2x^2}\right)\varphi_2^2 + \left(\frac{z^2}{x^2y^2}\right)\varphi_3^2 \\ & - 2\left[\frac{\varphi_1\varphi_2}{z^2} + \frac{\varphi_2\varphi_3}{x^2} + \frac{\varphi_3\varphi_1}{y^2}\right] = 0. \end{aligned} \quad (151)$$

Equation (151) can then be solved for $\varphi_3 = \varphi_3(\varphi_1, \varphi_2)$ and the remaining CDJ matrix elements determined. The result is given by

$$\begin{aligned} \varphi_3 &= \left(\frac{x}{z}\sqrt{\varphi_1} \pm \frac{y}{z}\sqrt{\varphi_2}\right)^2; \\ \Psi_1 &= -\left(\frac{y}{z}\right)\varphi_2 \pm \left(\frac{x}{z}\right)\sqrt{\varphi_1\varphi_2}; \\ \Psi_2 &= -\left(\frac{x}{z}\right)\varphi_1 \pm \left(\frac{y}{z}\right)\sqrt{\varphi_1\varphi_2}; \quad \Psi_3 = \pm\sqrt{\varphi_1\varphi_2}, \end{aligned} \quad (152)$$

whence Ψ_{ae} has been reduced to two physical D.O.F. While B_a^i and Ψ_{ae} are degenerate as is the 3-metric h_{ij} , the CDJ matrix still has a solution as in (152). One may attempt to determine the algebraic classification of the spacetime that this describes as in Paper VIII. The characteristic equation for the CDJ matrix is given by

$$\lambda^3 - (\text{tr}\Psi)\lambda^2 + \frac{1}{2}(\text{Var}\Psi)\lambda - \det\Psi = 0. \quad (153)$$

The invariants of Ψ_{ae} can be computed from (152), as well as what we have determined on the basis of the Hamiltonian constraint

$$\text{Var}\Psi = \det\Psi = 0; \quad \text{tr}\Psi = \varphi_1\left(1 + \frac{x}{z}\right) + \varphi_2\left(1 + \frac{y}{z}\right) \pm \frac{2xy}{z^2}\sqrt{\varphi_1\varphi_2}. \quad (154)$$

This reduces (153) to $\lambda^2(\lambda - \text{tr}\Psi) = 0$, with a double root $\lambda = 0$ and a single root $\lambda = \text{tr}\Psi$. A degenerate CDJ matrix with two equal eigenvalues implies a spacetime of Petrov type *III*.

4.4 Another extreme example

Let us now examine a potential with a different structure,

$$A_i^a = g \frac{x^a x^i}{r^3}. \quad (155)$$

The magnetic field for (155) has an abelian part

$$\begin{aligned} b_a^i &\equiv (B_a^i)_{Abelian} = g \epsilon^{ijk} \partial_j \left(\frac{x^a x^k}{r^3} \right) \\ &= g \epsilon^{ijk} \left(\frac{1}{r^3} (\delta_{aj} x^k + \delta_{jk} x^a) - \frac{3}{r^5} x^a x^k x^j \right) = g \epsilon^{iak} \frac{x^k}{r^3}, \end{aligned} \quad (156)$$

and a vanishing nonabelian part

$$(B_a^i)_{nonabelian} = \frac{1}{2} \epsilon^{ijk} f^{abc} \left(\frac{g x^b x^j}{r^3} \right) \left(\frac{g x^c x^i}{r^3} \right) = 0. \quad (157)$$

This is an example where $X_a \neq 0$ and $N_a = 0$, which by the previous analyses is integrable.¹⁶ The helicity density matrix C_{ae} also vanishes, since

$$C_{ae} = A_i^a B_e^i = \frac{g^2}{r^6} \epsilon^{iak} x^a x^i x^k = 0. \quad (158)$$

Note that A_i^a and B_a^i have switched roles in relation to the monopole example treated earlier. The Gauss' law vector fields with rescaled versions are given respectively by

$$\mathbf{v}_a = \frac{g}{r^3} \epsilon^{aij} x^i \frac{\partial}{\partial x^j}; \quad v_a = \epsilon^{aij} x^i \frac{\partial}{\partial x^j}. \quad (159)$$

These vector fields satisfy the commutator bracket $[v_a, v_b] = \epsilon_{abc} v_c$ for angular momentum, with numerically constant structure functions. The Gauss' Law constraint reduces to

$$\begin{pmatrix} v_2 & 0 & v_3 \\ v_1 & v_3 & 0 \\ 0 & v_2 & v_1 \end{pmatrix} \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = - \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

One may solve for $\Psi_f = \Psi_f[\vec{\varphi}]$ as

$$\vec{\Psi} = (1 + \pi)^{-1} v^{-1} M \vec{\varphi} = \hat{J}_f^g \varphi_g, \quad (160)$$

¹⁶The magnetic field is degenerate for this configuration, which makes the 3-metric h_{ij} also degenerate.

where we have defined

$$v = \begin{pmatrix} v_2 & 0 & 0 \\ 0 & v_3 & 0 \\ 0 & 0 & v_1 \end{pmatrix}; \quad \pi = \begin{pmatrix} \pi_2 & 0 & 0 \\ 0 & \pi_3 & 0 \\ 0 & 0 & \pi_1 \end{pmatrix}$$

as well as

$$M = \begin{pmatrix} 1 & v_3 v_1^{-1} v_2 v_3^{-1} & -v_3 v_1^{-1} \\ -v_1 v_2^{-1} & 1 & v_1 v_2^{-1} v_3 v_1^{-1} \\ v_2 v_3^{-1} v_1 v_2^{-1} & -v_2 v_3^{-1} & 1 \end{pmatrix}$$

The cyclic operators π_{x^i} are given by

$$\pi_2 = v_2^{-1} v_3 v_1^{-1} v_2 v_3^{-1} v_1; \quad \pi_3 = v_3^{-1} v_1 v_2^{-1} v_3 v_1^{-1} v_2; \quad \pi_1 = v_1^{-1} v_2 v_3^{-1} v_1 v_2^{-1} v_3 \quad (161)$$

With all the constituents of the inversion defined, we must define the action of v_a and v_a^{-1} in local coordinates. Since v_a generates rotations in the plane orthogonal to the coordinate x_a , we coordinatize the integral curves using angles ϕ_a , for $a = 1, 2, 3$, such that

$$v_a = \frac{\partial}{\partial \phi^a} \longrightarrow v_a^{-1} = \int d\phi_a. \quad (162)$$

Note that x^a is the spectator variable in (162). Along with the identifications (162) comes a prescription for changing into the appropriate coordinate system to carry out the inversion.

When acting with v_3 and v_3^{-1} , one performs the transformation $(x, y, z) \rightarrow (r, \phi_3, z)$ and its inverse, which leaves the z direction invariant.

$$\begin{aligned} x &= \sqrt{r^2 - z^2} \cos \phi_3; & y &= \sqrt{r^2 - z^2} \sin \phi_3; & z &= z; \\ r &= \sqrt{x^2 + y^2 + z^2}; & \phi_3 &= \tan^{-1}\left(\frac{y}{x}\right); & z &= z. \end{aligned} \quad (163)$$

When acting with v_1 and v_1^{-1} , one perform the transformation $(x, y, z) \rightarrow (r, \phi_1, z)$ and its inverse, which leaves the x direction invariant.

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \cos \phi_1; & z &= \sqrt{r^2 - x^2} \sin \phi_1; & x &= x; \\ r &= \sqrt{x^2 + y^2 + z^2}; & \phi_1 &= \tan^{-1}\left(\frac{z}{y}\right); & x &= x. \end{aligned} \quad (164)$$

When acting with v_2 and v_2^{-1} , one perform the transformation $(x, y, z) \rightarrow (r, \phi_2, z)$ and its inverse, which leaves the y direction invariant.

$$\begin{aligned}
z &= \sqrt{r^2 - y^2} \cos \phi_2; \quad x = \sqrt{r^2 - y^2} \sin \phi_2; \quad y = y; \\
r &= \sqrt{x^2 + y^2 + z^2}; \quad \phi_2 = \tan^{-1} \left(\frac{x}{z} \right); \quad y = y.
\end{aligned} \tag{165}$$

The end result is that the shear elements Ψ_f are images of anisotropy elements φ_f under coordinate transformations. Hence the latter play the role of the (three) physical degrees of freedom, with the former unphysical. In this example, both A_i^a and B_a^i are degenerate, which means that the spatial 3-metric h_{ij} will be degenerate in this case.

4.5 Instanton of nonzero size for $g = 2$

To provide a nondegenerate integrable example where C_{ae} is nonvanishing, let us consider the case of an instanton of size ρ and strength g . We will approach the problem in stages, first considering some simple cases and then moving on to the general case, to illustrate some techniques in integrability. Starting from a connection of

$$A_i^a = \frac{g}{r^2 + \rho^2} \epsilon_{aij} x^j, \quad (166)$$

the magnetic field and helicity density matrix are given by

$$B_a^i = \frac{1}{(r^2 + \rho^2)^2} (-2g\rho^2 \delta_{ai} + g(g-2)x^i x_a); \quad C_{ae} = -2 \frac{(\rho g)^2}{(r^2 + \rho^2)^3} \epsilon_{aej} x^j \quad (167)$$

For this configuration the Gauss' law constraint is given by

$$\begin{aligned} & \frac{2g\rho^2}{(r^2 + \rho^2)^2} \left[\begin{pmatrix} v_y & 0 & v_z \\ v_x & v_z & 0 \\ 0 & v_y & v_x \end{pmatrix} + \frac{g}{r^2 + \rho^2} \begin{pmatrix} 3y & 0 & 3z \\ 3x & 3z & 0 \\ 0 & 3y & 3x \end{pmatrix} \right] \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} \\ & + \frac{2g\rho^2}{(r^2 + \rho^2)^2} \left[\begin{pmatrix} v_x & 0 & 0 \\ 0 & v_y & 0 \\ 0 & 0 & v_z \end{pmatrix} + \frac{g}{r^2 + \rho^2} \begin{pmatrix} 2x & -x & -x \\ -y & 2y & -y \\ -z & -z & 2z \end{pmatrix} \right] \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0, \end{aligned}$$

where we have defined

$$v_y = \frac{\partial}{\partial y} - \left(\frac{g-2}{2\rho^2} \right) y \frac{\partial}{\partial t}; \quad \frac{\partial}{\partial t} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}, \quad (168)$$

and similarly for v_x and v_z .¹⁷ The differential operators satisfy the algebra

$$[\partial_t, \partial_{x^i}] = -\partial_{x^i} \quad \forall i. \quad (169)$$

We will first treat the $g = 2$ case, since the contribution to (168) proportional to ∂_t vanishes which simplifies the inversion. We will retain the symbol g in what follows, with the understanding that $g = 2$. Note that the helicity density matrix C_{ae} can be combined with the spatial gradient terms using an integrating factor

¹⁷Recall from previously that ∂_t is integrable.

$$\left(\frac{\partial}{\partial x} + \frac{ngx}{r^2 + \rho^2}\right)\Psi = (r^2 + \rho^2)^{-ng/2} \frac{\partial}{\partial x} \left((r^2 + \rho^2)^{ng/2} \Psi\right). \quad (170)$$

This is possible since $C_{11} = C_{22} = C_{33} = 0$, which corresponds to one of the off-diagonal helicity configurations, whence (25) reduces to just the first term. For $g = 2$, the Gauss' law constraint reduces to

$$\begin{aligned} & (r^2 + \rho^2)^{-3g/2} \begin{pmatrix} \partial_y & 0 & \partial_z \\ \partial_x & \partial_z & 0 \\ 0 & \partial_y & \partial_x \end{pmatrix} (r^2 + \rho^2)^{3g/2} \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} \\ &= -(r^2 + \rho^2)^{-g} \begin{pmatrix} \partial_x & -x & -x \\ -y & \partial_y & -y \\ -z & -z & \partial_z \end{pmatrix} (r^2 + \rho^2)^g \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}. \end{aligned}$$

Then the solution is given by

$$\Psi_g = \hat{J}_g^f \varphi_f, \quad (171)$$

where the Gauss' law propagator is explicitly given by

$$\hat{J}_g^f = -(r^2 + \rho^2)^{-3g/2} \begin{pmatrix} \partial_y & 0 & \partial_z \\ \partial_x & \partial_z & 0 \\ 0 & \partial_y & \partial_x \end{pmatrix}^{-1} (r^2 + \rho^2)^{g/2} \begin{pmatrix} \partial_x & -x & -x \\ -y & \partial_y & -y \\ -z & -z & \partial_z \end{pmatrix} (r^2 + \rho^2)^g.$$

The operator ordering must be preserved, but it is straightforward to carry out the operations, which have been reduced to computing the inverse

$$\begin{pmatrix} \partial_y & 0 & \partial_z \\ \partial_x & \partial_z & 0 \\ 0 & \partial_y & \partial_x \end{pmatrix}^{-1} = (1 + \pi)^{-1} v^{-1} M.$$

Using the results from the previous section on inversion, one defines

$$v = \begin{pmatrix} \partial_y & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_x \end{pmatrix}; \quad \pi = \begin{pmatrix} \pi_y & 0 & 0 \\ 0 & \pi_z & 0 \\ 0 & 0 & \pi_x \end{pmatrix}$$

as well as

$$M = \begin{pmatrix} 1 & \partial_z \partial_x^{-1} \partial_y \partial_z^{-1} & -\partial_z \partial_x^{-1} \\ -\partial_x \partial_y^{-1} & 1 & \partial_x \partial_y^{-1} \partial_z \partial_x^{-1} \\ \partial_y \partial_z^{-1} \partial_x \partial_y^{-1} & -\partial_y \partial_z^{-1} & 1 \end{pmatrix}$$

The cyclic operators π_{x^i} are given by

$$\begin{aligned}
\pi_y &= \partial_y^{-1} \partial_z \partial_x^{-1} \partial_y \partial_z^{-1} \partial_x; \\
\pi_z &= \partial_z^{-1} \partial_x \partial_y^{-1} \partial_z \partial_x^{-1} \partial_y; \\
\pi_x &= \partial_x^{-1} \partial_y \partial_z^{-1} \partial_x \partial_y^{-1} \partial_z.
\end{aligned} \tag{172}$$

If one restricts oneself to integrations where the integration constants are set to zero, then the operators ∂_x and ∂_x^{-1} are inverses of each other. All operations in the cyclic operators cancel, yielding $\pi_x \equiv \pi_y \equiv \pi_z \equiv 1$ which is the identity operator on functions, implying that $(1 + \pi)^{-1} = \frac{1}{2}$. Analogous cancellations occur in M , and we are led to the following result

$$\begin{aligned}
v_{ae}^{-1} &= \begin{pmatrix} \partial_y & 0 & \partial_z \\ \partial_x & \partial_z & 0 \\ 0 & \partial_y & \partial_x \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} \partial_y^{-1} & 0 & 0 \\ 0 & \partial_z^{-1} & 0 \\ 0 & 0 & \partial_x^{-1} \end{pmatrix} \begin{pmatrix} 1 & \partial_x^{-1} \partial_y & -\partial_z \partial_x^{-1} \\ -\partial_x \partial_y^{-1} & 1 & \partial_y^{-1} \partial_z \\ \partial_z^{-1} \partial_x & -\partial_y \partial_z^{-1} & 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \partial_y^{-1} & \partial_x^{-1} & -\partial_y^{-1} \partial_z \partial_x^{-1} \\ -\partial_z^{-1} \partial_x \partial_y^{-1} & \partial_z^{-1} & \partial_y^{-1} \\ \partial_z^{-1} & -\partial_x^{-1} \partial_y \partial_z^{-1} & \partial_x^{-1} \end{pmatrix}.
\end{aligned}$$

One can verify by the multiplication of the operator matrices, that $v_{ae}^{-1} v_{ef} = \delta_{af}$. The end result for $g = 2$ is that one can define the Gauss' law propagator \hat{T}_g^f as an operator-valued matrix, such that

$$\Psi_f = \hat{T}_f^g \varphi_g, \tag{173}$$

where

$$\begin{aligned}
\hat{T}_g^f &= -(r^2 + \rho^2)^{-3g/2} \begin{pmatrix} \partial_y^{-1} & \partial_x^{-1} & -\partial_y^{-1} \partial_z \partial_x^{-1} \\ -\partial_z^{-1} \partial_x \partial_y^{-1} & \partial_z^{-1} & \partial_y^{-1} \\ \partial_z^{-1} & -\partial_x^{-1} \partial_y \partial_z^{-1} & \partial_x^{-1} \end{pmatrix} \\
&\quad (r^2 + \rho^2)^{g/2} \begin{pmatrix} \partial_x & -x & -x \\ -y & \partial_y & -y \\ -z & -z & \partial_z \end{pmatrix} (r^2 + \rho^2)^g,
\end{aligned}$$

The operator ordering must be strictly maintained in this inversion. According to (173), the shear elements Ψ_f are a map of the anisotropy elements φ_f under the action of \hat{T}_f^g . There is a reduction in Ψ_{ae} by three D.O.F., consequently we for the purposes of this paper we treat φ_f as the physical degrees of freedom. The action is well-defined, as one can verify by multiplication of the matrix elements. The result is that the metric for an instanton on nonzero size should be nondegenerate.

We have shown that the Gauss' law constraint has a well-defined solution for $g = 2$ for the instanton, but we would like to extend it to more general values of g . One possibility is to modify (173) to

$$\Psi_f = \hat{T}_f^g \varphi_g - \rho_f^g \Psi_g, \quad (174)$$

where we have defined

$$\rho_f^g = \left(\frac{2-g}{2\rho^2} \right) \begin{pmatrix} y & 0 & z \\ x & z & 0 \\ 0 & y & x \end{pmatrix} \frac{\partial}{\partial t},$$

which parametrizes the deviation from $g = 2$. The operator ∂_t has a simple action on polynomials, and for $A_N(\vec{r}) \in H_n$, the space of homogeneous polynomials, given by

$$A_N(\vec{r}) = \sum_{l+m+n=N} A_{lmn} x^l y^m z^n, \quad (175)$$

we have that

$$\frac{\partial}{\partial t} A_N(x) = N A_N(x); \quad \partial_t^{-1} A_N(x) = \frac{1}{N} A_N(x). \quad (176)$$

The general solution to the Gauss' law constraint for this configuration is given by

$$\Psi_f = \hat{J}_f^g \varphi_g, \quad (177)$$

where we have defined the Gauss' law propagator \hat{J}_f^g by the infinite series

$$\hat{J}_f^g = (\delta_f^g + \rho_f^g)^{-1} = \delta_f^g + \sum_{n=1}^{\infty} (-1)^n \rho_f^{f_1} \rho_{f_1}^{f_2} \dots \rho_{f_{n-1}}^{f_n} \rho_{f_n}^g. \quad (178)$$

A necessary condition for convergence of (178) is that $|g - 2| < 2\rho^2$.

We can get a better idea of the action of the differential operators by explicitly evaluating their action on a set of basis functions, first considering the effect without the magnetic helicity density. Take v_x for example, where we wish to solve the differential equation

$$\left(\frac{\partial}{\partial x} - kx \frac{\partial}{\partial t} \right) \Psi = Q. \quad (179)$$

Equation (179) yields

$$\Psi = \left(\frac{1}{1 - k \int dx x \frac{\partial}{\partial t}} \right) Q, \quad (180)$$

which upon choosing Q to be any single term of the form (176) expands into the infinite series

$$\Psi = \sum_N \left(k \int dx x \frac{\partial}{\partial t} \right)^N (x^l y^m z^n). \quad (181)$$

The zeroth order action is the identity. The first order term is given by

$$k \int dx x \frac{\partial}{\partial t} (x^l y^m z^n) = k(l+m+n) \int dx (x^{l+1} y^m z^n) = kx^2 \left(\frac{l+m+n}{l+2} \right) (x^l y^m z^n) \quad (182)$$

The repeated action leads to the following series

$$\begin{aligned} \Psi = x^l y^m z^n & \left[1 + k \left(\frac{l+m+n}{l+2} \right) x^2 + k^2 \left(\frac{l+m+n}{l+2} \right) \left(\frac{l+m+n+2}{l+4} \right) \right. \\ & \left. + k^3 \left(\frac{l+m+n}{l+2} \right) \left(\frac{l+m+n+2}{l+4} \right) \left(\frac{l+m+n+4}{l+6} \right) x^6 + \dots \right]. \quad (183) \end{aligned}$$

This is a hypergeometric series

$$\Psi = x^l y^m z^n {}_1F_1 \left(0, \frac{l+m+n}{2}; \frac{l}{2}; kx^2 \right), \quad (184)$$

and the ratio test yields a radius of convergence

$$-\sqrt{\frac{1}{k}} < x < \sqrt{\frac{1}{k}}, \quad k = \frac{g-2}{2\rho^2}. \quad (185)$$

Performing the analogue of (180) for the y and z directions enables one to construct the Gauss' law propagators \hat{J}_f^g and the 'deformed' basis elements T_{ae}^g . Provided that the physical degrees of freedom $\vec{\varphi}$ are restricted to the basis functions (176), then one may solve the Hamiltonian constraint (132) with respect to this basis. Since our main purpose in this paper is to demonstrate the integrability of the Gauss' law constraint, we will not carry out this exercise here.

4.6 Instanton of nonzero size for $g \neq 2$

We will now treat the general case of the previous example, using the general technique of integral curves. First note that the magnetic field can be written in the form

$$B_a^i = \frac{1}{(r^2 + \rho^2)^2} \begin{pmatrix} 1 + (mx)^2 & m^2xy & m^2xz \\ m^2yx & 1 + (my)^2 & m^2yz \\ m^2zx & m^2zy & 1 + (mz)^2 \end{pmatrix},$$

where we have defined the mass scale

$$m = \frac{1}{\rho} \sqrt{\frac{2-g}{2}}. \quad (186)$$

We can directly read off the vector fields as

$$\begin{aligned} w_x &= (1 + (mx)^2) \frac{\partial}{\partial x} + (mx) \left(y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) + \frac{3g}{\rho^2 + r^2}; \\ w_y &= (1 + (my)^2) \frac{\partial}{\partial y} + (my) \left(z \frac{\partial}{\partial z} + x \frac{\partial}{\partial x} \right) + \frac{3g}{\rho^2 + r^2}; \\ w_z &= (1 + (mz)^2) \frac{\partial}{\partial z} + (mz) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + \frac{3g}{\rho^2 + r^2}. \end{aligned} \quad (187)$$

Since B_a^i is conformally related to a magnetic field of the correct form, this means that the system is integrable in terms of coordinates. We will illustrate for the x direction with the remaining cases to follow by cyclic permutation. We will take the integral curves as $\vec{\gamma} = (u, v, w)$. From (187) we read off the following relations

$$\frac{dx}{du} = 1 + (mx)^2; \quad \frac{dy}{du} = (mx)y; \quad \frac{dz}{du} = (mx)z. \quad (188)$$

The first equation of (188) integrates to

$$u - u_0 = \int \frac{dx}{1 + (mx)^2} \longrightarrow x = \frac{1}{m} \tan[\tan^{-1}(mx_0) + m(u - u_0)], \quad (189)$$

which defines the projection of the integral curve u onto the x direction. To find the projections onto the y and z directions we must substitute (189) into the second and third equations of (188). For the y direction this yields

$$\frac{dy}{y} = (mx) du = \tan[\tan^{-1}(mx_0) + m(u - u_0)] du, \quad (190)$$

which restricts the x dependence to the curve u , integrating to

$$y = \frac{y_0}{\cos[\tan^{-1}(mx_0) + m(u - u_0)]}; \quad z = \frac{z_0}{\cos[\tan^{-1}(mx_0) + m(u - u_0)]} \quad (191)$$

We have included the analogous steps for the y and z directions.

Let us define the following functions

$$\begin{aligned} \eta(u; x_0) &= \tan[\tan^{-1}(mx_0) + m(u - u_0)]; \\ \zeta(u; x_0) &= \cos[\tan^{-1}(mx_0) + m(u - u_0)]. \end{aligned} \quad (192)$$

Then the integral curves $\vec{\gamma}$ are well-defined,¹⁸ and given by

$$x(u) = \frac{\eta(u; x_0)}{m}; \quad y(u) = \frac{y_0}{\zeta(u; x_0)}; \quad z(u) = \frac{z_0}{\zeta(u; x_0)} \quad (193)$$

for the u curve,

$$x(v) = \frac{x_0}{\zeta(v; y_0)}; \quad y(v) = \frac{\eta(v; y_0)}{m}; \quad z(v) = \frac{z_0}{\zeta(v; y_0)} \quad (194)$$

for the v curve, and

$$x(w) = \frac{x_0}{\zeta(w; z_0)}; \quad y(w) = \frac{y_0}{\zeta(w; z_0)}; \quad z(w) = \frac{\eta(w; z_0)}{m} \quad (195)$$

for the w curve. Having constructed the integral curves of the vector fields v_a , we must now take into account the helicity density contributions. First, note that (187) simplifies to

$$w_x = \frac{d}{du} - \frac{g}{\rho^2 + r^2(u)}; \quad w_y = \frac{d}{dv} - \frac{g}{\rho^2 + r^2(v)}; \quad w_z = \frac{d}{dw} - \frac{g}{\rho^2 + r^2(w)} \quad (196)$$

We can now treat incorporate the helicity density contributions as integrating factors, restricted to the appropriate integral curves. Make the following definitions for integrating factors

$$\begin{aligned} \phi_u &= \exp \left[g \int_{u_0}^u \left[\rho^2 + \left(\frac{\eta(u'; x_0)}{m} \right)^2 + \left(\frac{y_0}{\zeta(u'; x_0)} \right)^2 + \left(\frac{z_0}{\zeta(u'; x_0)} \right)^2 \right]^{-1} du' \right]; \\ \phi_v &= \exp \left[g \int_{v_0}^v \left[\rho^2 + \left(\frac{\eta(v'; y_0)}{m} \right)^2 + \left(\frac{z_0}{\zeta(v'; y_0)} \right)^2 + \left(\frac{x_0}{\zeta(v'; y_0)} \right)^2 \right]^{-1} dv' \right]; \\ \phi_w &= \exp \left[g \int_{w_0}^w \left[\rho^2 + \left(\frac{\eta(w'; z_0)}{m} \right)^2 + \left(\frac{x_0}{\zeta(w'; z_0)} \right)^2 + \left(\frac{y_0}{\zeta(w'; z_0)} \right)^2 \right]^{-1} dw' \right] \end{aligned} \quad (197)$$

¹⁸The parts where η and ζ blow up correspond to infinite values of x , y and z , therefore these integral curves should cover Σ within a finite range of the parameters of the curves.

While the results of the integrations are unwieldy to display in closed form, the point is that the integrating factors are path independent and well-defined for $g \leq 2$.

We can now write the inverse Gauss' law operator on the shear subspace, rigorously, as

$$\mathbf{w}_{ae}^{-1} = \begin{pmatrix} \phi_v \circ \partial_v \circ \phi_v^{-1} & 0 & \phi_w \circ \partial_w \circ \phi_w^{-1} \\ \phi_u \circ \partial_u \circ \phi_u^{-1} & \phi_w \circ \partial_w \circ \phi_w^{-1} & 0 \\ 0 & \phi_v \circ \partial_v \circ \phi_v^{-1} & \phi_u \circ \partial_u \circ \phi_u^{-1} \end{pmatrix}^{-1} = (1+\pi)^{-1} \mathbf{w}^{-1} M$$

where we have defined

$$M = \begin{pmatrix} 1 & \phi_u \circ \partial_u^{-1} \circ \phi_{uv} \circ \partial_v \circ \phi_v^{-1} & -\phi_w \circ \partial_w \circ \phi_{wu} \circ \partial_u^{-1} \circ \phi_u^{-1} \\ -\phi_u \circ \partial_u \circ \phi_{uv} \circ \partial_v^{-1} \circ \phi_v^{-1} & 1 & \phi_v \circ \partial_v^{-1} \circ \phi_{vw} \circ \partial_w \circ \phi_w^{-1} \\ \phi_w \circ \partial_w^{-1} \circ \phi_{wu} \circ \partial_u \circ \phi_u^{-1} & -\phi_v \circ \partial_v \circ \phi_{vw} \circ \partial_w^{-1} \circ \phi_w^{-1} & 1 \end{pmatrix}$$

with $\phi_{uv} = \phi^{-1} \circ \phi_v$, and

$$\mathbf{w} = \begin{pmatrix} \phi_v \circ \partial_v \circ \phi_v^{-1} & 0 & 0 \\ 0 & \phi_w \circ \partial_w \circ \phi_w^{-1} & 0 \\ 0 & 0 & \phi_u \circ \partial_u \circ \phi_u^{-1} \end{pmatrix}; \quad \pi = \begin{pmatrix} \pi_2 & 0 & 0 \\ 0 & \pi_3 & 0 \\ 0 & 0 & \pi_1 \end{pmatrix}$$

as well as the cyclic operators

$$\pi_2 = \pi_3 = \pi_1 = 1. \tag{198}$$

4.7 Another nondegenerate example

For a potential of the form $A_i^a = \delta_{ai}(g/r)$ for $r > 0$, the magnetic field is given by

$$B_a^i = \delta_{ai} \left(\frac{g}{r} \right)^2 + \epsilon_{iaj} x^j \left(\frac{g}{r^3} \right), \quad (199)$$

and the helicity density matrix by

$$C_{ae} = A_e^a B_e^i = \delta_{ae} \left(\frac{g}{r} \right)^3 + \epsilon_{aej} x^j \left(\frac{g^2}{r^4} \right), \quad (200)$$

which is an off-diagonal helicity configuration. The abelian part of B_a^i yields a vector field

$$X_a = B_a^i \partial_i = \left(\frac{g}{r^3} \right) \epsilon_{aij} x^i \frac{\partial}{\partial x^j} = \left(\frac{g}{r^3} \right) J_a, \quad (201)$$

where J_a are the orbital angular momentum operators, which satisfy the Lie algebra $[J_a, J_b] = i\epsilon_{abc} J_c$. The integral curves of J_i correspond to a flow

$$\phi_{\theta_i}(x^i, x^j, x^k) = (x^i, x^j \cos \theta_i - x^k \sin \theta_i, x^k \cos \theta_i + x^j \sin \theta_i), \quad (202)$$

which leaves x^i invariants with $(x^i, x^j, x^k) = \text{Perm}(x, y, z)$. Note that J_i annihilates functions of r as well as functions of the single variable x^i . The nonabelian part of \mathbf{v}_a is given by

$$N_a = \left(\frac{g}{r} \right)^2 \frac{\partial}{\partial x^a}. \quad (203)$$

The Gauss' law constraint is given by

$$\begin{aligned} & \left(\frac{g}{r} \right)^2 \left[\begin{pmatrix} v_y & 0 & v_z \\ v_x & v_z & 0 \\ 0 & v_y & v_x \end{pmatrix} + \frac{3}{r^2} \begin{pmatrix} y & 0 & z \\ x & z & 0 \\ 0 & y & x \end{pmatrix} \right] \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} \\ & + \left(\frac{g}{r^2} \right) \left[\begin{pmatrix} v_x & 0 & 0 \\ 0 & v_y & 0 \\ 0 & 0 & v_z \end{pmatrix} + \frac{1}{r^2} \begin{pmatrix} 2x & -x & -x \\ -y & 2y & -y \\ -z & -z & 2z \end{pmatrix} \right] \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0, \end{aligned}$$

where we have defined $v_x = X_x + N_x$, with the Abelian and nonabelian contributions are given, up to a factor of $(g/r)^2$ by

$$X_x = \left(\frac{g}{r}\right)J_x; \quad N_x = \partial_x \quad (204)$$

and similarly for v_y and v_z .

Since C_{ae} is an off-diagonal helicity configuration, the Gauss' law constraint can be written

$$\begin{pmatrix} V_y & 0 & V_z \\ V_x & V_z & 0 \\ 0 & V_y & V_x \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} + \begin{pmatrix} V_x - \frac{x}{r^2} & -\frac{x}{r^2} & -\frac{x}{r^2} \\ -\frac{y}{r^2} & V_y - \frac{y}{r^2} & -\frac{y}{r^2} \\ -\frac{z}{r^2} & -\frac{z}{r^2} & V_z - \frac{z}{r^2} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0$$

where we have defined

$$V_x = \frac{\partial}{\partial x} + \frac{3x}{r^2} + \frac{J_x}{rg} \quad (205)$$

and likewise for V_y and V_z . The solution is formally given by

$$\Psi_g = \hat{J}_g^f \varphi_f, \quad (206)$$

where we have defined the Gauss' law propagator

$$\hat{J}_g^f = - \begin{pmatrix} V_y & 0 & V_z \\ V_x & V_z & 0 \\ 0 & V_y & V_x \end{pmatrix}^{-1} \begin{pmatrix} V_x - \frac{x}{r^2} & -\frac{x}{r^2} & -\frac{x}{r^2} \\ -\frac{y}{r^2} & V_y - \frac{y}{r^2} & -\frac{y}{r^2} \\ -\frac{z}{r^2} & -\frac{z}{r^2} & V_z - \frac{z}{r^2} \end{pmatrix}$$

for the chosen configuration. Using the results from the previous section on inversion, one can explicitly write the inverse

$$V_{ae}^{-1} = \begin{pmatrix} V_y & 0 & V_z \\ V_x & V_z & 0 \\ 0 & V_y & V_x \end{pmatrix}^{-1} = (1 + \pi)^{-1} v^{-1} M,$$

where the constituent parts are given by

$$M = \begin{pmatrix} 1 & V_z V_x^{-1} V_y V_z^{-1} & -V_z V_x^{-1} \\ -V_x V_y^{-1} & 1 & V_x V_y^{-1} V_z V_x^{-1} \\ V_y V_z^{-1} V_x V_y^{-1} & -V_y V_z^{-1} & 1 \end{pmatrix}$$

and

$$v = \begin{pmatrix} V_y & 0 & 0 \\ 0 & V_z & 0 \\ 0 & 0 & V_x \end{pmatrix}; \quad \pi = \begin{pmatrix} \pi_y & 0 & 0 \\ 0 & \pi_z & 0 \\ 0 & 0 & \pi_x \end{pmatrix}$$

as well as the cyclic operators

$$\begin{aligned}\pi_y &= V_y^{-1} V_z V_x^{-1} V_y V_z^{-1} V_x; \\ \pi_z &= V_z^{-1} V_x V_y^{-1} V_z V_x^{-1} V_y; \\ \pi_x &= V_x^{-1} V_y V_z^{-1} V_x V_y^{-1} V_z.\end{aligned}\tag{207}$$

To give this all meaning, one must understand the action of $V_{x^i}^{-1}$. Let us demonstrate with the x direction for illustrative purposes. To find the action of V_x^{-1} is the same as attempting to solve the equation

$$V_x\{\Psi\} \equiv \left(\frac{\partial}{\partial x} + \frac{3x}{r^2} + \frac{J_x}{rg}\right)\Psi(x, y, z) = Q(x, y, z)\tag{208}$$

for Ψ in terms of Q , given Ψ as the image of Q under some map. Note that $J_x = y\partial_z - z\partial_y$ has a trivial action on the x and the r dependence of operations, since

$$J_x\{x\} = J_y\{y\} = J_z\{z\} = J_x\{r\} = J_y\{r\} = J_z\{r\} = 0.\tag{209}$$

We may exploit this observation to construct an integrating factor operator corresponding to each direction

$$\phi_x = r^3(2(r+x))^{\frac{1}{g}J_x}; \quad \phi_y = r^3(2(r+y))^{\frac{1}{g}J_y}; \quad \phi_z = r^3(2(r+z))^{\frac{1}{g}J_z},\tag{210}$$

where the coordinates are rescaled to make them dimensionless. This transforms (208) into the form

$$\phi_x^{-1}\partial_x(\psi_x\phi) = Q \longrightarrow \Psi = \phi_x^{-1} \int dx \phi_x Q(x, y, z) \Big|_{y,z},\tag{211}$$

which integrates the x dependence while holding y and z constant, and similarly for the other directions. It is convenient to think of this as a composition

$$V_x \equiv \phi_x^{-1} \circ \partial_x \circ \phi_x; \quad V_x^{-1} \equiv \phi_x^{-1} \circ \partial_x^{-1} \circ \phi_x,\tag{212}$$

taking care to preserve the operator ordering. Then ϕ_x is nothing other than a homeomorphism from Σ onto a particular chart adapted to operations in the x direction. Define transition function operators $U_{x^i x_j} = \phi_{x^i} \phi_{x_j}^{-1}$. Then the cyclic operators (207) are given by

$$\begin{aligned}
\pi_y &= \phi_y^{-1} \circ \partial_y^{-1} \circ U_{yz} \circ \partial_z \circ U_{zx} \circ \partial_x^{-1} \circ U_{xy} \circ \partial_y \circ U_{yz} \circ \partial_z^{-1} \circ U_{zx} \circ \partial_x \circ \phi_x \\
\pi_z &= \phi_z^{-1} \circ \partial_z^{-1} \circ U_{zx} \circ \partial_x \circ U_{xy} \circ \partial_y^{-1} \circ U_{yz} \circ \partial_z \circ U_{zx} \circ \partial_x^{-1} \circ U_{xy} \circ \partial_y \circ \phi_y \\
\pi_x &= \phi_x^{-1} \circ \partial_x^{-1} \circ U_{xy} \circ \partial_y \circ U_{yz} \circ \partial_z^{-1} \circ U_{zx} \circ \partial_x \circ U_{xy} \circ \partial_y^{-1} \circ U_{yz} \circ \partial_z \circ \phi_z
\end{aligned} \tag{213}$$

If not for the the transition operators U_{xy} , the cyclic operators π_x , π_y and π_z would be the identity. Indeed in the strong coupling limit $g \rightarrow \infty$, we have that

$$\lim_{g \rightarrow \infty} \phi_{x^i} = \lim_{g \rightarrow \infty} U_{x^i x^j} = 1. \tag{214}$$

So the transition functions $U_{x^i x^j} = U_{x^i x^j}(g)$ are parametrized by g .¹⁹ The matrix M is can be written as

$$M = \begin{pmatrix} 1 & V_y \circ \pi_y \circ V_x^{-1} & -V_z V_x^{-1} \\ -V_x V_y^{-1} & 1 & V_z \circ \pi_z \circ V_y^{-1} \\ V_x \circ \pi_x \circ V_z^{-1} & -V_y V_z^{-1} & 1 \end{pmatrix}$$

or in component form taking into account the composition operators

$$\begin{aligned}
M_{ae} &= \delta_{ae} - \delta_{ay} \delta_{ex} \phi_x^{-1} \circ \partial_x \circ U_{xy} \circ \partial_y^{-1} \circ \phi_y \\
&+ \delta_{ax} \delta_{ey} \phi_y^{-1} \circ \partial_y \circ \phi_y \circ \pi_y \circ \phi_x^{-1} \circ \partial_x^{-1} \circ \phi_x + \dots,
\end{aligned} \tag{215}$$

where the dots cycle through the cyclic permutations of indices (x, y, z) . Hence we have provided a prescription for inverting the Gauss' law constraint for this configuration, which leaves remaining the Hamiltonian constraint to fix the φ_f . Once this is completed, then the 3-metric can be constructed from the CDJ matrix and the magnetic field B_a^i from (199), as in

$$h_{ij} = (\det \Psi[\vec{\varphi}]) (\hat{T}_{ab}^g \hat{T}_{bf}^{g'} \varphi_g \varphi_{g'})^{-1} (B^{-1})_i^a (B^{-1})_j^f, \tag{216}$$

where we have defined

$$\hat{T}_{ae}^g \equiv e_{ae}^g + E_{ae}^f \hat{J}_f^g. \tag{217}$$

¹⁹In the large g limit the computation of \hat{J}_g^f proceeds straightforwardly as in the previous subsection, so we do not repeat it here. One could envision performing an asymptotic expansion in inverse powers of g to find the Gauss' law propagator for large $g < \infty$.

4.8 Diagonal connection

Finally, we will consider a diagonal configuration with three degrees of freedom per point, for example one of the quantizable instanton representation configurations identified in Paper XIII. The connection is given by

$$A_i^a = \begin{pmatrix} A_1^1 & 0 & 0 \\ 0 & A_2^2 & 0 \\ 0 & 0 & A_3^3 \end{pmatrix}$$

This yields a magnetic field and helicity density matrix, respectively, of

$$B_a^i = \begin{pmatrix} A_2^2 A_3^3 & -\partial_z A_2^2 & \partial_y A_3^3 \\ \partial_z A_1^1 & A_3^3 A_1^1 & -\partial_x A_3^3 \\ -\partial_y A_1^1 & \partial_x A_2^2 & A_1^1 A_2^2 \end{pmatrix}; \quad C_{ae} = \begin{pmatrix} A_1^1 A_2^2 A_3^3 & -A_1^1 \partial_z A_2^2 & A_1^1 \partial_y A_3^3 \\ A_2^2 \partial_z A_1^1 & A_2^2 A_3^3 A_1^1 & -A_2^2 \partial_x A_3^3 \\ -A_3^3 \partial_y A_1^1 & A_3^3 \partial_x A_2^2 & A_3^3 A_1^1 A_2^2 \end{pmatrix}.$$

We have chosen x , y and z for coordinates, without loss of generality since these are merely labels. The vector fields \mathbf{v}_a are given by

$$\begin{aligned} \mathbf{v}_1 &= A_2^2 A_3^3 \frac{\partial}{\partial x} + \left(\frac{\partial A_1^1}{\partial y} \right) \frac{\partial}{\partial z} - \left(\frac{\partial A_1^1}{\partial z} \right) \frac{\partial}{\partial y}; \\ \mathbf{v}_2 &= A_3^3 A_1^1 \frac{\partial}{\partial y} + \left(\frac{\partial A_2^2}{\partial z} \right) \frac{\partial}{\partial x} - \left(\frac{\partial A_2^2}{\partial x} \right) \frac{\partial}{\partial z}; \\ \mathbf{v}_3 &= A_1^1 A_2^2 \frac{\partial}{\partial z} + \left(\frac{\partial A_3^3}{\partial x} \right) \frac{\partial}{\partial y} - \left(\frac{\partial A_3^3}{\partial y} \right) \frac{\partial}{\partial x}. \end{aligned} \quad (218)$$

We see the familiar decomposition of \mathbf{v}_a into a part containing a Hamiltonian vector field X_a and a part containing a nonabelian contribution N_a . In the analysis of the previous section we have treated X_a and N_a as separate, delineating the condition for each part individually. But we will show that the present example is an example where the two vector fields can be combined into one.

For \mathbf{v}_1 the Hamiltonian function is A_1^1 , which induces a flow in the two dimensional hyperplane dual to the x direction according to the Hamiltonian equations

$$\frac{dy}{du} = \frac{\partial A_1^1}{\partial z}; \quad \frac{dz}{du} = -\frac{\partial A_1^1}{\partial y}. \quad (219)$$

Since A_1^1 is assumed to be a smooth function of x , y and z , then there exists a Hamiltonian flow generated along a one dimensional curve $\gamma_x(y, z)$

$$y = y(y_0, z_0; u) \Big|_x; \quad z = z(y_0, z_0; u) \Big|_x, \quad (220)$$

in the (y, z) plane for each fixed x , where u parametrizes location along the curve. More precisely, (220) is the projection of the congruence $\vec{\gamma}$ onto the (y, z) plane. To obtain the three dimensional picture, let us ‘unfreeze’ the x dependence and set

$$\frac{dx}{du} = A_2^2 A_3^3 = f(y_0, z_0; x, u). \quad (221)$$

We have expressed A_2^2 and A_3^3 in a chart adapted to u , which entails substitution of (220) into the y and z dependence of these functions. We must invert (221) along $\vec{\gamma}$, which entails an antidifferentiation with respect to the parameter u . If $\vec{\gamma}$ forms a congruence, then this should be possible since there are no self-intersections of the integral curves of \mathbf{v}_a by supposition.²⁰ Equation (221) is an ODE, which we should expect to have a unique solution. A way to see this is using Pickard’s theorem in the theory of exterior differential systems. Namely, given the one form $\theta^1 = dx - f(x, u)du \in R^2$ with coordinates (x, u) , then there exists a one-to-one correspondence between solutions to (221) and curves $\gamma : R \rightarrow R^2$ such that $\vec{\gamma}(\theta^1) = 0$ and $\vec{\gamma}(du) \neq 0$. The result is that restricted to the curve $\vec{\gamma}$, the vector field \mathbf{v}_2 is integrable. Another way to see this is that

$$d\theta^1 = -\partial_x f dx \wedge du \longrightarrow \theta^1 \wedge d\theta^1 = 0, \quad (222)$$

namely that the topological torsion of θ^1 vanishes when restricted to γ_u .

Applying a similar analysis to \mathbf{v}_2 we have

$$\frac{dz}{dv} = \frac{\partial A_2^2}{\partial x}; \quad \frac{dx}{dv} = -\frac{\partial A_2^2}{\partial z}. \quad (223)$$

Since A_2^2 is assumed to be a smooth function of x , y and z , then there exists a Hamiltonian flow generated along a one dimensional curve $\gamma_y(z, x)$

$$z = y(z_0, x_0; u) \Big|_y; \quad x = x(z_0, x_0; v) \Big|_x, \quad (224)$$

in the (z, x) plane for each fixed y , where v parametrizes location along the curve. Hence, (223) is the projection of the congruence $\vec{\gamma}$ onto the (z, x) plane. To obtain the three dimensional picture, we ‘unfreeze’ the y dependence and set

$$\frac{dy}{dv} = A_3^3 A_1^1 = g(z_0, x_0; y, v). \quad (225)$$

²⁰This should be the case if the integral curves are topologically equivalent to the Cartesian axes, namely derivable from them by continuous deformation.

Equation (225) implies the vanishing $\gamma_v(\boldsymbol{\theta}^2) = 0$ of the one form $\boldsymbol{\theta}^2$ restricted to the integral curve γ_v , where $\boldsymbol{\theta}^2 = dy - g(y, v)dv$. Likewise, the topological torsion vanishes along this curve

$$d\boldsymbol{\theta}^2 = -\partial_y g dy \wedge dv \longrightarrow \boldsymbol{\theta}^2 \wedge d\boldsymbol{\theta}^2 = 0. \quad (226)$$

Applying a similar analysis to \mathbf{v}_3 , we have

$$\frac{dx}{dw} = \frac{\partial A_3^3}{\partial y}, \quad \frac{dy}{dw} = -\frac{\partial A_3^3}{\partial x}. \quad (227)$$

Since A_3^3 is assumed to be a smooth function of x , y and z , then there exists a Hamiltonian flow generated along a one dimensional curve $\gamma_z(x, y)$

$$x = x(x_0, y_0; w) \Big|_z; \quad y = y(x_0, y_0; w) \Big|_z, \quad (228)$$

in the (x, y) plane for each fixed z , where w parametrizes location along the curve. Hence, (228) is the projection of the congruence $\vec{\gamma}$ onto the (x, y) plane. To obtain the three dimensional picture, we ‘unfreeze’ the z dependence and set

$$\frac{dz}{dw} = A_1^1 A_2^2 = h(x_0, y_0; z, w). \quad (229)$$

Equation (229) implies the vanishing $\gamma_w(\boldsymbol{\theta}^3) = 0$ of the one form $\boldsymbol{\theta}^3$ restricted to the integral curve γ_w , where $\boldsymbol{\theta}^3 = dz - g(y, v)dw$. Likewise, the topological torsion vanishes along this curve

$$d\boldsymbol{\theta}^3 = -\partial_z h dz \wedge dw \longrightarrow \boldsymbol{\theta}^3 \wedge d\boldsymbol{\theta}^3 = 0. \quad (230)$$

We have seen, as a consequence of the Hamiltonian flow induced by the abelian part of the vector fields, that the vector fields are actually invertible including the nonabelian part, when one restricts the inversion to the integral curves $\vec{\gamma} = (\gamma_u, \gamma_v, \gamma_w)$. Hence assuming that $\vec{\gamma}$ is topologically equivalent to the Cartesian coordinate system, then we have

$$\begin{aligned} \mathbf{v}_1^{-1}\{\Psi(u, v, w)\} &= \int du \Psi(u, v, w) \Big|_{v, w \text{ const.}}; \\ \mathbf{v}_2^{-1}\{\Psi(u, v, w)\} &= \int dv \Psi(u, v, w) \Big|_{w, u \text{ const.}}; \\ \mathbf{v}_3^{-1}\{\Psi(u, v, w)\} &= \int dw \Psi(u, v, w) \Big|_{u, v \text{ const.}} \end{aligned} \quad (231)$$

for a smooth function Ψ . Equation (231) is an abuse of notation, since Ψ may not initially be expressed in the chart (u, v, w) . More precisely we must perform the composition of each operation with the conversion into the appropriate chart as in

$$\mathbf{v}_1 \equiv \phi_u^{-1} \circ \frac{d}{du} \circ \phi_u; \quad \mathbf{v}_2 \equiv \phi_v^{-1} \circ \frac{d}{dv} \circ \phi_v; \quad \mathbf{v}_3 \equiv \phi_w^{-1} \circ \frac{d}{dw} \circ \phi_w. \quad (232)$$

We will take these ϕ operations to be implied, and revert to the former abuse of notation for convenience.

Having put in place the aforementioned structures, we can now solve the Gauss' law constraint for the quantizable configurations. This is given by matrix form by

$$\begin{pmatrix} \frac{d}{dv} + c_{13}(v) & 0 & \frac{d}{dw} + c_{12}(w) \\ \frac{d}{du} + c_{23}(u) & \frac{d}{dw} + c_{21}(w) & 0 \\ 0 & \frac{d}{dv} + c_{31}(v) & \frac{d}{du} + c_{32}(u) \end{pmatrix} \begin{pmatrix} \Psi_3 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} \\ = - \begin{pmatrix} \frac{d}{du} + \partial_x(A_2^2 A_3^3) & -A_3^3 \partial_x A_2^2 & -A_2^2 \partial_x A_3^3 \\ -A_3^3 \partial_y A_1^1 & \frac{d}{dv} + \partial_y(A_3^3 A_1^1) & -A_1^1 \partial_y A_3^3 \\ -A_2^2 \partial_z A_1^1 & -A_1^1 \partial_z A_2^2 & \frac{d}{dw} + \partial_z(A_1^1 A_2^2) \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix},$$

where we have defined

$$\begin{aligned} c_{13} &= A_1^1 \partial_y A_3^3 + 2A_3^3 \partial_y A_1^1; & c_{12} &= A_1^1 \partial_y A_2^2 + 2A_2^2 \partial_y A_1^1; \\ c_{23} &= A_2^2 \partial_y A_3^3 + 2A_3^3 \partial_z A_2^2; & c_{21} &= A_2^2 \partial_z A_1^1 + 2A_1^1 \partial_z A_2^2; \\ c_{31} &= A_3^3 \partial_y A_1^1 + 2A_1^1 \partial_y A_3^3; & c_{32} &= A_3^3 \partial_x A_2^2 + 2A_2^2 \partial_x A_3^3 \end{aligned} \quad (233)$$

adapted to the appropriate parameters for the given integral curves. The main point that this is an off-diagonal helicity configuration, which reduces the inversion of the Gauss' law constraint to the inversion of operators of the form

$$\begin{aligned} \mathbf{w}_1 &= \frac{d}{du} + c(u) = e^{-\int c(u)du} \frac{d}{du} e^{\int c(u)du}; \\ \mathbf{w}_2 &= \frac{d}{dv} + c(v) = e^{-\int c(v)dv} \frac{d}{dv} e^{\int c(v)dv}; \\ \mathbf{w}_3 &= \frac{d}{dw} + c(w) = e^{-\int c(w)dw} \frac{d}{dw} e^{\int c(w)dw}. \end{aligned} \quad (234)$$

The point is that restricted to the corresponding integral curves, one can write down integrating factors in the appropriate chart. The inversion of the Gauss' constraint then takes place exactly as for the off-diagonal helicity configurations.

5 Inversion of the vector fields in the general case

We have seen that the Gauss' law constraint is exactly integrable for a large class of configurations for which X_a and N_a are directly related, which establishes a well-defined map $\vec{\varphi} \rightarrow \vec{\Psi}[\vec{\varphi}]$. But we also would like to establish the map for the general case of configurations A_i^a .²¹ The solution to the Gauss' law constraint requires a choice of $B_a^i \in C^\infty(\Sigma)$ combined with a well-defined prescription for carrying out the inversion. Whatever the prescription chosen, it is necessary to evaluate the inverses \mathbf{v}_1^{-1} , \mathbf{v}_2^{-1} and \mathbf{v}_3^{-1} as part of the inversion process. Assuming that this is well-defined then one ideally would like to integrate along the \mathbf{v}_a direction using holonomic coordinates, which requires the projection of \mathbf{v}_a onto the axes of a specified holonomic coordinate system.

Let us say that we wish to invert the vector field \mathbf{v}_a . First start with the differential equation

$$\mathbf{v}_a \Psi(x^i, x^j, x^k) = Q(x^i, x^j, x^k), \quad (235)$$

where $\Psi(x^i, x^j, x^k) \in \text{Dom}\{\mathbf{v}_a\}$ and $Q(x^i, x^j, x^k) \in \text{Ran}\{\mathbf{v}_a\}$ and $i \neq j \neq k$ ²² and the vector field \mathbf{v}_a is given by²³

$$\mathbf{v}_a = B_a^i \frac{\partial}{\partial x^i} + B_a^j \frac{\partial}{\partial x^j} + B_a^k \frac{\partial}{\partial x^k} \quad (236)$$

Since \mathbf{v}_a might not necessarily be holonomic, we must specify a holonomic coordinate x^k and then project \mathbf{v}_a onto the x^k direction. This choice corresponds to the specification of a particular element $B_a^i \equiv B_a^{x^i}$ which will be divided out to normalize \mathbf{v}_a so that the integration can be carried out in this direction. Define the ratios

$$\beta_{ik}^a = \frac{B_a^i}{B_a^k}; \quad \beta_{jk}^a = \frac{B_a^j}{B_a^k} \quad (237)$$

where $B_a^k \neq 0$.²⁴ Hence, (235) reduces to

²¹The purpose of the Gauss' law constraint is to establish the map from the physical degrees of freedom $\vec{\varphi}$, irrespective of the well-definedness of the shear elements.

²²The following argument can be generalized to all possibilities by cyclic permutation of indices.

²³We will often abstain from using the Einstein summation convention, since we would like to explicitly depict the relation of the coordinate being manipulated to the spectator coordinates. Hence the notation (x^i, x^j, x^k) will correspond to a given permutation of the Cartesian coordinates (x, y, z) which are holonomic. The indices (i, j, k) each label a specific coordinate, as opposed to a range of coordinates.

²⁴Note, since it might not be possible to satisfy this condition globally in Σ , one is free to choose a different $B_a^i \neq 0$ and then redo the procedure.

$$\left(\frac{\partial}{\partial x^k} + \beta_{ik}^a(x^i, x^j, x^k) \frac{\partial}{\partial x^i} + \beta_{jk}^a(x^i, x^j, x^k) \frac{\partial}{\partial x^j}\right) \Psi = \frac{Q(x^i, x^j, x^k)}{B_a^k(x^i, x^j, x^k)}. \quad (238)$$

Next, act with the inverse differential operator corresponding to x^k . Hence $(\partial_x)^{-1} \equiv \int dx$ and similarly for y and for z . By integrating both sides of (238) in the x^k direction, we obtain

$$\begin{aligned} \left[1 + \int_{\xi_0}^{x^k} d\xi \left(\beta_{ik}^a(x^i, x^j, \xi) \frac{\partial}{\partial x^i} + \beta_{jk}^a(x^i, x^j, \xi) \frac{\partial}{\partial x^j}\right)\right] \Psi(x^i, x^j, \xi) \\ = \Psi(x^i, x^j, \xi_0) + \int_{\xi_0}^{x^k} d\xi \frac{Q(x^i, x^j, \xi)}{B_a^k(x^i, x^j, \xi)} \end{aligned} \quad (239)$$

where $\Psi(x^i, x^j, \xi_0) \in Ker\{\partial/\partial x^k\}$ is the boundary data, which must be specified on the $x^k = \xi_0$ hyperplane. The solution then is given by

$$\begin{aligned} \Psi(x^i, x^j, x^k) &= \hat{K}_k^a(x^i, x^j; x^k, \xi_0) \Psi(x^i, x^j, \xi_0) \\ &+ \hat{K}_k^a(x^i, x^j; x^k, \xi') \int_{\xi_0}^{\xi'} d\xi'' \frac{Q(x^i, x^j, \xi'')}{B_a^k(x^i, x^j, \xi'')}, \end{aligned} \quad (240)$$

where we have defined a parallel propagator with respect to the x^k direction, given by

$$\hat{K}_k^a(x^i, x^j; x^k, \xi') = \hat{P} \left(\exp \left[- \int_{\xi'}^{x^k} d\xi'' \left(\beta_{ik}^a(x^i, x^j, \xi'') \frac{\partial}{\partial x^i} + \beta_{jk}^a(x^i, x^j, \xi'') \frac{\partial}{\partial x^j} \right) \right] \right). \quad (241)$$

Here in (241), \hat{P} corresponds to path ordering along a path projected onto the x^k direction, the direction of integration in the inversion of the vector field \mathbf{v}_a . For example when integrating in the z direction, the action of \mathbf{v}_a^{-1} is given by

$$\begin{aligned} \Psi(x, y, z) &= \mathbf{v}_a^{-1} Q(x, y, z) \\ &= \exp \left(- \int_0^z d\xi \left(\beta_{xz}^a(x, y, \xi) \frac{\partial}{\partial x} + \beta_{yz}^a(x, y, \xi) \frac{\partial}{\partial y} \right) \right) \\ &\quad \left[\Psi(x, y, 0) + \int_0^z d\xi' \frac{Q(x, y, \xi')}{B_a^z(x, y, \xi')} \right]. \end{aligned} \quad (242)$$

We have omitted the path-ordering symbol in (235), which should be understood to act on both the exponential as well as the integral in the third line.

5.1 Gauss' law dual translation operators

It will be convenient, for a certain class of configurations, to interpret the inverted Gauss' law operators as translation operators restricted to a two dimensional spatial hyperplane dual to a particular direction of path-ordered integration. The Baker–Campbell–Hausdorff formula for two operators A and B is given by []

$$\exp(A)\exp(B) = \exp\mu(A, B), \quad (243)$$

where $\mu(A, B)$ is given by

$$\begin{aligned} \mu(A, B) = & A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \frac{1}{12}[B, [B, A]] \\ & + \frac{1}{24}[B, [B, [B, A]]] + \frac{1}{24}[A, [A, [A, B]]] + \dots \end{aligned} \quad (244)$$

The left hand side of (243) presents two disentangled operators A and B , which the right hand side combines into a single operator $C = \mu(A, B)$. The decomposition into A and B is in general not unique, and may in general take on an intricate form due to noncommutativity of the operators. However for a given operator A , the operator B is uniquely determined once C is specified.

To illustrate, let us express \hat{K}_k^a in the form (243)

$$\hat{K}_k^a(x^i, x^j; x^k, \xi') = \exp\left[-\int dx^k b_{ik}^a(x^i, x^j, x^k) \frac{\partial}{\partial x^i}\right] \exp\left[-\int dx^k b_{jk}^a(x^i, x^j, x^k) \frac{\partial}{\partial x^j}\right] \quad (245)$$

where b_{ik}^a and b_{jk}^a must be determined. The quantities β_{ik}^a and β_{jk}^a in (241) are determined by B_a^i , which fixes b_{ik}^a and b_{jk}^a .²⁵ For the operator ordering chosen in (245), the action of \hat{K}_k^a is that of a sequence of translations in Σ by predetermined magnetic field-dependent amounts. Hence,

$$\hat{K}_k^a Q(x^i, x^j, x^k) = Q\left(x^i - \int dx^k b_{ik}^a(x^i, x^j, x^k), x^j - \int dx^k b_{jk}^a(x^i, x^j, x^k), x^k\right) \quad (246)$$

with cyclic permutation of indices, where we use the indefinite integration operator to signify integration in the x^k direction. Note that the x^k argument of Q is unaffected by this integration, since that particular x^k dependence is annihilated by the $\partial/\partial x^i$ and $\partial/\partial x^j$ derivatives. The effect is to

²⁵One imposes the condition that the left hand side of (245) decomposes into the right hand side with the operators $\partial/\partial x^i$ and $\partial/\partial x^j$ disentangled, and then use this condition to find the coefficients b_{ik}^a and b_{jk}^a . Alternatively, one may first choose b_{ik}^a and b_{jk}^a , using this to define the β_{ik}^a and β_{jk}^a , which in turn fix the ratios of the Ashtekar magnetic field components.

translate the source Q according to b_{ik}^a and b_{jk}^a , which are related to β_{ik}^a . The β_{jk}^a are specified by choosing a well-defined configuration for B_a^i , then the procedure must be repeated for each B_a^i . Note that the dual translations in the (x^i, x^j) hyperplane commute with integration normal to the plane, hence (246).²⁶

The procedure to invert the Gauss' law covariant differential operators is then as follows. First the practitioner must choose a configuration, namely a specified function $B_a^i = B_a^i(\vec{r}) \in C^\infty(\Sigma)$, and then evaluate the functional form of $\Psi_{ae} = \Psi_{ae}[\vec{B}_b(\vec{r})]$ for that particular configuration, for each configuration.²⁷ One chooses the six functions $b_{ik}^a(\vec{r})$ and $b_{jk}^a(\vec{r})$, which define a flow lines along which the source Q_a becomes translated. One may then construct the Ashtekar magnetic field by first computing β from b using (243),(244),²⁸ hence the parametrization

$$B_a^i = \begin{pmatrix} B_1^3 & 0 & 0 \\ 0 & B_2^3 & 0 \\ 0 & 0 & B_3^3 \end{pmatrix} \begin{pmatrix} \beta_{13}^1[b] & \beta_{23}^1[b] & 1 \\ \beta_{13}^2[b] & \beta_{23}^2[b] & 1 \\ \beta_{13}^3[b] & \beta_{23}^3[b] & 1 \end{pmatrix}$$

corresponding to an integration in the $x^k = z$ direction.²⁹ The Ashtekar magnetic field B_a^i defines three directions in Σ , one direction for each a . Hence, provided the Gauss' law constraint is satisfied for $\vec{\theta} = \vec{\theta}[\vec{\lambda}, \vec{B}_a]$, then the resulting Ashtekar curvature $F_{jk}^a = \epsilon_{ijk} B_a^i$ constitutes a general solution to the Einstein equations by construction.³⁰ The physical interpretation is that B_a^i defines the vector fields \mathbf{v}_a , the flow of which uniquely specifies a congruence of curves along which the solution to the Gauss' law constraint can be constructed. The action of \mathbf{v}_a^{-1} on any function $Q(x_i, x_j, x_k)$ is given by³¹

²⁶The interpretation in terms of translation operators will notably facilitate the proof of convergence of the solutions, since for $|Q| < \infty \forall \vec{r} \in \Sigma$, one still has that $|\hat{K}_e Q| < \infty \forall \vec{r} \in \Sigma$ for appropriate choice of Q .

²⁷One must additionally have specified the functions corresponding to two eigenvalues $\lambda_1(\vec{r})$ and $\lambda_2(\vec{r})$ of the CDJ matrix Ψ_{ae} , with the third eigenvalue $\lambda_3(\vec{r})$ fixed by the Hamiltonian constraint.

²⁸Assuming the convergence of the series (243), these two characterizations are equivalent. We assume convergence based on the Baker–Campbell–Hausdorff theorem, which guarantees the existence of an element C of the algebra, such that $e^A e^B = e^C$.

²⁹Once a direction of integration has chosen and the boundary data specified, then the integration must be carried out without exception along that direction.

³⁰This is due to the CDJ Ansatz, and holds for algebraically general spacetimes

³¹Not worrying about the covariant-contravariant position of the indices

$$\begin{aligned}
\Phi(x_i, x_j, x_k) &= \mathbf{v}_a^{-1} Q(x_i, x_j, x_k) \\
&= \exp \left[- \int_0^{x_k} d\xi' \left(\beta_{ik}^a(x_i, x_j, \xi') \frac{\partial}{\partial x_i} + \beta_{jk}^a(x_i, x_j, \xi') \frac{\partial}{\partial x_j} \right) \right] \Phi(x_i, x_j, 0) \\
&+ \exp \left[- \int_{\xi'}^{x_k} d\xi'' \left(\beta_{ik}^a(x_i, x_j, \xi'') \frac{\partial}{\partial x_i} + \beta_{jk}^a(x_i, x_j, \xi'') \frac{\partial}{\partial x_j} \right) \right] \int_0^{\xi'} d\xi \frac{Q(x_i, y_i, \xi)}{B_a^k(x_i, y_i, \xi)}. \\
&= \Phi \left(x_i - \int_0^{x_k} d\xi \beta_{ik}^a(x_i, x_j, \xi), x_j - \int_0^{x_k} d\xi \beta_{jk}^a(x_i, x_j, \xi), 0 \right) \\
&+ \int_0^{x_k} d\xi \frac{Q \left(x_i - \int_0^\xi d\xi' \beta_{ik}^a(x_i, x_j, \xi'), x_j - \int_0^\xi d\xi' \beta_{jk}^a(x_i, x_j, \xi'), \xi \right)}{B_a^k \left(x_i - \int_0^\xi d\xi' \beta_{ik}^a(x_i, x_j, \xi'), x_j - \int_0^\xi d\xi' \beta_{jk}^a(x_i, x_j, \xi'), \xi \right)}.
\end{aligned} \tag{247}$$

5.2 Criteria for boundedness of the inversion

It is not enough to prove convergence of the individual operators \mathbf{v}_a^{-1} when inverting the Gauss' law constraint. One must additionally take into account the effect of the insertions α_{ae} in (25), which also come into play in placing bounds upon the norm $|\Psi_a|$. This in turn depends upon choice of B_a^i and the CDJ matrix eigenvalues (λ_1, λ_2) , which are chosen $\forall x \in \Sigma$ in advance of solving the constraint, and the source term Q_a as well as the boundary data β_a . We prove well-behaved generic choices of $\lambda_1 = \lambda_1(\vec{r})$ and $\lambda_2 = \lambda_2(\vec{r})$ and $B_a^i = B_a^i(\vec{r})$ place an upper bound on the solution.

First, we require that for $f = (\lambda_1, \lambda_2, \lambda_3, B_a^i)$ and their derivatives exist everywhere $f(\vec{r}) \in C^\infty(\Sigma) < \infty \forall \vec{r} \in \Sigma$. The following norms can be defined $\|f\| = \max\{|f(\vec{r})|\} \forall \vec{r} \in \Sigma$ and $\|\beta\| = \max\{|\beta(\vec{r})|\} \forall \vec{r} \in \Sigma$. The zeroth order term of (25) is given by

$$G_0 = \beta(x, y, 0) + K \int_0^y \frac{Q(x, y', z)}{B_y^a(x, y', z)} dy' = \beta + K \int_0^y f(\vec{r}). \tag{248}$$

The effect of the dual translation operator K does not affect the bound on $|f|$ since it serves to translate the source within its domain of definition, which is chosen to be well-defined $\forall x \in \Sigma$ based upon the configuration chosen. We require the boundary data to be similarly well-defined. Hence, one has that

$$|G_0| \leq \|\beta\| + \|f\| \left| \int_0^y dy' \right| = \|\beta\| + \|f\| y. \tag{249}$$

The first order term is given by³²

³²There is no loss of generality by using the direction y , which is provided for clarity of

$$G_1 = K \int_0^y dy' \frac{\alpha(x, y', z)}{B_k^a(x, y', z)} \beta(x, 0, z) + K \int_0^y dy' \frac{\alpha(x, y', z)}{B_k^a(x, y', z)} \int_0^{y'} \frac{Q(x, y'', z)}{B_a^k(x, y'', z)} dy'' \quad (250)$$

Define the ratio

$$\frac{\alpha(\vec{r})}{B_a^k(\vec{r})} \equiv c(\vec{r}) \quad (251)$$

and the following norm $\|c\| = \max\{|c(\vec{r})|\} \forall \vec{r} \in \Sigma$.³³ Then the second term of (250) is bounded by

$$\int_0^y dy' |c(y')| |G_0(y')| \leq \int_0^y dy' \|c\| \|f\| y' = \frac{y^2}{2} \|c\| \|f\|. \quad (252)$$

Then the following inequality holds

$$|G_1| \leq \|c\| \|\beta\| y + \frac{y^2}{2} \|c\| \|f\|. \quad (253)$$

One finds the following recursion relation for the $(n+1)^{th}$ term

$$|G_{n+1}| \leq \|\beta\| + \int_0^y dy' |c(y')| |G_n(y')|, \quad (254)$$

which leads to the following inequality

$$|G_{n+1}| \leq \|\beta\| \frac{\|c\|^n y^n}{n!} + \frac{\|f\|}{\|c\|} \left(\frac{\|c\|^{n+1} y^{n+1}}{(n+1)!} \right). \quad (255)$$

Application of the Minkowski inequality to (25), where each successive term is bounded by (255), one has

$$\begin{aligned} |(\mathbf{v} + \alpha)^{-1} Q| &\leq \sum_{n=0}^{\infty} \|\beta\| \frac{\|c\|^n y^n}{n!} + \frac{\|f\|}{\|c\|} \sum_{n=0}^{\infty} \frac{\|c\|^{n+1} y^{n+1}}{(n+1)!} \\ &= \|\beta\| e^{\|c\| y} + \frac{\|f\|}{\|c\|} \left(e^{\|c\| y} - 1 \right). \end{aligned} \quad (256)$$

the exposition. $|y| = \max\{|x|, |y|, |z|\}$ simply plays the role of placing an upper bound on each term, which will in turn place an upper bound on the norm of (25).

³³As with the previously defined norms, $\|c\|$ is also completely known in advance of carrying out the inversion based upon the specified configuration.

Hence, given the norms

$$\begin{aligned}
\| f \| &= \max \left| \frac{Q}{B_a^i} \right| \quad \forall i, a \quad \forall \vec{r} \in \Sigma \text{ and } B_a^i \neq 0; \\
\| c \| &= \max \left| \frac{\alpha_{ae}}{B_b^j} \right| \quad \forall a, e, i, b \quad \forall \vec{r} \in \Sigma \text{ and } B_a^i \neq 0; \\
y[\vec{r}] &= \max\{|x|, |y|, |z|\} \tag{257}
\end{aligned}$$

where $\vec{r} = (x, y, z)$ in the last line of (257) corresponds to a particular point in Σ at which the constraint, is being solved, one has that

$$|(\mathbf{v} + \alpha)^{-1}Q| < \infty \tag{258}$$

or that the full series (25) is bounded. Boundedness follows from the observation that the norm of (25) is bounded by a pointwise convergent series (256). Note that boundedness for an infinite series does not imply convergence of the series.

6 Solution to the initial value constraints by expansion about the Kodama state

Let us now demonstrate a method to solve the constraints when there is a nonzero cosmological constant Λ . It is known that the Kodama state ψ_{Kod} solves the constraints, which corresponds to a CDJ matrix $\Psi_{ae} = -\frac{3}{\Lambda}\delta_{ae}$. This automatically satisfies the diffeomorphism constraint

$$\epsilon_{ijk}B_a^jB_e^k\Psi_{ae} = -\frac{6}{\Lambda}\epsilon_{ijk}B_a^jB_e^k\delta_{ae} = 0, \quad (259)$$

due to antisymmetry of ϵ_{ijk} . The Gauss' law constraint is given by

$$\mathbf{w}_e(\Psi_{ae}) = B_e^i \frac{\partial \Psi_{ae}}{\partial x^i} + C_{fe}(f_{afg}\Psi_{ge} + f_{efg}\Psi_{ag}). \quad (260)$$

For $\Psi_{ae} = -\frac{6}{\Lambda}\delta_{ae}$ we have

$$\begin{aligned} & B_e^i \partial_i \Psi_{ae} + B_e^i A_i^f (f_{afg}\Psi_{ge} + f_{efg}\Psi_{ag}) \\ &= B_e^i \partial_i \delta_{ae} + B_e^i A_i^f (f_{afg}\delta_{ge} + f_{efg}\delta_{ag}) \\ &= 0 + B_e^i A_i^f (f_{afe} + f_{efa}) = 0 \end{aligned} \quad (261)$$

due to antisymmetry of the structure constants f_{abc} . The Hamiltonian constraint yields

$$(Var\Psi + \Lambda \det\Psi) \Big|_{\Psi_{ae} = -(3/\Lambda)\delta_{ae}} = 0. \quad (262)$$

The existence of an exact solution to the constraints motivates the search for other solutions by perturbative methods. So we will construct additional solutions by expansion about the solution $\Psi_{ae} = -\frac{3}{\Lambda}\delta_{ae}$. Define the relation

$$\Psi_{ae} = -\left(\frac{3}{\Lambda}\delta_{ae} + \epsilon_{ae}\right), \quad (263)$$

where ϵ_{ae} parametrizes fluctuations about the conformally self-dual sector of Einstein manifolds³⁴ Substitution of (263) into the Gauss' law and diffeomorphism constraints yields a cancellation of the $-\frac{6}{\Lambda}\delta_{ae}$ term, and these constraints can be rewritten by replacement of Ψ_{ae} with ϵ_{ae} to yield

$$\epsilon_{ijk}B_a^jB_e^k\epsilon_{ae}0; \quad \mathbf{w}_e\{\epsilon_{ae}\} = 0. \quad (264)$$

³⁴The includes S^4 and CP_2 [?].

The first equation of (264) implies that $\epsilon_{ae} = \epsilon_{(ae)}$ is symmetric in ae , whence ϵ_{ae} is now given by

$$\epsilon_{ae} = e_{ae}^f \epsilon_f + E_{ae}^f \Psi_f. \quad (265)$$

Note that the $-\frac{3}{\Lambda}\delta_{ae}$ term in (263) has no effect on the off-diagonal terms in (265). The Gauss' law constraint reduces to

$$\Psi_f = \hat{J}_f^{f'} \epsilon_{f'} \quad (266)$$

where \hat{J}_g^f is the Gauss' law propagator, which establishes a map from the anisotropy to the shear elements.³⁵ Upon substitution of (263) into the Hamiltonian constraint and cancelling off a factor of Λ one obtains the Hamiltonian constraint in terms of $\epsilon = \epsilon_{ae}$, given by

$$H = \text{tr} \epsilon + \frac{\Lambda}{3} \text{Var} \epsilon + \frac{\Lambda^2}{3} \det \epsilon = 0. \quad (267)$$

Unlike for the kinematic constraints, there exist remnants of the cosmological constant Λ in (267) since the Hamiltonian constraint is nonlinear in Ψ_{ae} . Observe that (267) involves only the invariants of ϵ_{ae} , which can be written explicitly in terms of the eigenvalues (ϕ_1, ϕ_2, ϕ_3) as

$$\phi_1 + \phi_2 + \phi_3 + \frac{2\Lambda}{3}(\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1) + \frac{\Lambda^2}{3}\phi_1\phi_2\phi_3 = 0. \quad (268)$$

One obtains a closed form solution for ϕ_3 in terms of ϕ_1 and ϕ_2 ,

$$\phi_3 = -\left(\frac{\phi_1 + \phi_2 + \frac{\Lambda}{3}\phi_1\phi_2}{1 + \frac{2\Lambda}{3}(\phi_1 + \phi_2) + \frac{\Lambda^2}{3}\phi_1\phi_2}\right), \quad (269)$$

whence any reference to the Gauss' law constraint does not explicitly appear. Since the CDJ matrix is symmetric according to (264), when diagonalizable it can be written as a polar decomposition

$$\Psi_{bf} = -(e^{\theta \cdot T})_{ba'} \left(\frac{3}{\Lambda} \delta_{a'e'} + e_{a'e'}^1 \phi_1 + e_{a'e'}^2 \phi_2 + e_{a'e'}^3 \phi_3(\phi_1, \phi_2) \right) (e^{-\theta \cdot T})_{e'f} \quad (270)$$

where $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$ are a triple of complex $SO(3, C)$ rotation parameters. While (270) encodes the Hamiltonian constraint solution within ϕ_1 and ϕ_2 ,

³⁵Thus the shear elements are not independent degrees of freedom, but simply an image of the anisotropy elements.

regarded as the physical degrees of freedom, it does not incorporate the Gauss' law constraint G_a . In Paper VI it is shown how G_a can be used to fix the parameters $\vec{\theta}$ in the polar representation of the initial value constraints.

On the other hand, when one attempts to solve the Hamiltonian constraint in the Cartesian representation, there enter unphysical degrees of freedom. The Hamiltonian constraint (267) can be written as

$$H = \delta^{ae} \epsilon_{ae} + \Lambda I^{abef} \epsilon_{ae} \epsilon_{bf} + \Lambda^2 E^{abefcg} \epsilon_{ae} \epsilon_{bf} \epsilon_{cg} \quad (271)$$

where we have defined the $SO(3, C)$ invariant tensors

$$I^{abef} = \frac{1}{3}(\delta^{ae} \delta^{bf} - \delta^{eb} \delta^{af}); \quad E^{abefcg} = \frac{1}{18} \epsilon^{abc} \epsilon^{efg} \quad (272)$$

on the 6-dimensional Euclidean space of of CDJ deviation matrix elements ϵ_{ae} . Upon implementation of the Gauss' law constraint one projects onto the 3-dimensional subspace of anisotropy elements ϵ_e using

$$\epsilon_{ae} = (e_{ae}^g + E_{ae}^f \hat{J}_f^g) \epsilon_g \equiv T_{ae}^f[A] \epsilon_f. \quad (273)$$

One obtains a corresponding projection of the invariant tensors (272) onto corresponding vectors

$$Q^{f'g'} \equiv I^{abef} T_{ae}^{f'} T_{bf}^{g'}, \quad Q^{f'g'h'} \equiv E^{abefcg} T_{ae}^{f'} T_{bf}^{g'} T_{cg}^{h'}. \quad (274)$$

Making the identification of the diagonal elements $\epsilon_{ff} \equiv \epsilon_f$, we can project the Hamiltonian constraint onto the reduced space as

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + Q[\vec{B}_a; \epsilon_1, \epsilon_2, \epsilon_3] = 0, \quad (275)$$

which consists of a linearized part with a nonlinear error term Q , given by

$$Q_\Lambda = \Lambda Q^{fg} \epsilon_f \epsilon_g + \Lambda^2 Q^{fgh} \epsilon_f \epsilon_g \epsilon_h = Q_\Lambda[\epsilon; A]. \quad (276)$$

There are a few things to note regarding (275). First, it applies equally to the full theory as it does to the spatially homogeneous sector of A_i^a . Additionally, it enables one to solve the constraints by fixed point iteration, by expansion in powers of the cosmological constant Λ for each configuration A_i^a . Hence, one defines a recursion $(\epsilon_3)_{(n)}$, where $(\epsilon_3)_{(0)} = -\epsilon_1 - \epsilon_2$ is the zeroth order solution. Then one defines the recursion relation

$$(\epsilon_3)_{(n+1)} = -\epsilon_1 - \epsilon_2 - Q_\Lambda[\epsilon_1, \epsilon_2, (\epsilon_3)_{(n)}; A]. \quad (277)$$

The full solution is given by $\epsilon_3 = \lim_{n \rightarrow \infty} (\epsilon_3)_{(n)}$, which can be written as

$$\Psi_{ae} = -\left(\frac{3}{\Lambda}\delta_{ae} + T_{ae}^1\epsilon_1 + T_{ae}^2\epsilon_2 + T_{ae}^3\epsilon_3[\epsilon_1, \epsilon_2; A]\right) \quad (278)$$

Equations (278) signifies that the physical D.O.F. in the Cartesian representation are given by ϵ_1 and ϵ_2 , with the Gauss' law constraint encoded in the curved basis T_{ae}^f .³⁶ Consistency of (278) with (270) implies the following relation

$$T_{ae}^f\epsilon_f = (e^{\theta \cdot T})_{aa'}(e^{\theta \cdot T})_{ee'}e_{a'e'}^f\varphi_f, \quad (279)$$

which relates the anisotropy elements directly to the eigenvalues. It is shown in Paper VI how (279) can be used to find the $SO(3, C)$ angles $\vec{\theta}$.

³⁶Note that the only part of the configuration variables in the homogeneous sector that enter into this expansion are e_1 , e_2 and e_3 , which involve projections of $(A^{-1})_i^a$ onto the direction of k_i . The effect is to contract the spatial indices, which leaves remaining only internal indices. This may be interpreted as fixing a gauge which breaks the diffeomorphism invariance, by singling out a special direction \vec{k} in Σ .

7 Discussion

In this paper we have placed the configuration space Ashtekar/instanton representation variables into a new context. It turns out in the latter variables that the initial value constraints can be regarded as constraints on the momentum space of $\Omega_{Inst} = (\Psi_{ae}, A_i^a)$. When one resudes this space to its physical degrees of freedom, this leaves remaining the question of the canonically conjugate variables on the full space. The association of the connection with a $SO(3, C)$ frame and a choice of coordinates seems natural and it extends to the magnetic field B_a^i . On a more basic level we take the integral curves $\vec{\gamma}$ from whose vector fields these configuration variables can be derived as the fundamental quantities. Then each congruence determines an equivalence class of configurations on which the Gauss' law constraint may be evaluated. It then makes sense to regard the momentum space variables on the constraint shell as being aspects of the instanton representation which are independent of coordinates and of the $SO(3, C)$ frame. We have performed a pre-cursory analysed various configurations based upon the integrability of the vector fields, developing intuition on certain available configurations. Once one finds an integrable configuration, there seems to be no obstruction to pairing it with different values for the CDJ matrix. We have found a large class of configurations for which the Gauss' law constraint is integrable. This provides a large class of configurations for which one can in principle construct solutions to the Einstein equations according to the prescription of Paper II.

8 Appendix A: Expansion of the helicity density matrix in the tensor representation

We would like to write the Gauss' law constraint in a form more amenable to solution, in conjunction with the Hamiltonian and the diffeomorphism constraints, to the initial value problem of general relativity. Rewriting the constraint for completeness, we have

$$\mathbf{w}_e(\Psi_{ae}) \equiv \mathbf{w}_a^{fg}(\Psi_{fg}) = \delta^{af} \mathbf{v}_g(\Psi_{fg}) + C_a^{fg} \Psi_{fg} = GQ_a. \quad (280)$$

To compute the matrix elements of the connection C_a^{fg} we will need to evaluate the components of the matrix product $C_{be} = B_e^i A_i^b$ in the tensor representation.³⁷ The $SU(2) \otimes SU(2)$ -valued matrix C_{be} constitutes the connection corresponding to the $SU(2)_-$ Gauss' law covariant derivative of the CDJ deviation matrix elements. The tensor representation of the connection for this covariant derivative is given by

$$(C_a)^{fg} = C_{be} (f_{abf} \delta_{ge} + f_{ebg} \delta_{af}) \quad (281)$$

where the structure constants are given by $f_{abc} = \epsilon_{abc}$. We will now explicitly compute the components.

Starting with the 'even' elements corresponding to the first component of the $SU(2)_-$ charge

$$\begin{aligned} (C_1)^{12} &= C_{be} (\epsilon_{1b1} \delta_{2e} + \epsilon_{eb2} \delta_{11}) = C_{13} - C_{31} = -C_{[31]}; \\ (C_1)^{23} &= C_{be} (\epsilon_{1b2} \delta_{3e} + \epsilon_{eb3} \delta_{12}) = C_{33} \epsilon_{132} = -C_{33}; \\ (C_1)^{31} &= C_{be} (\epsilon_{1b3} \delta_{1e} + \epsilon_{eb1} \delta_{13}) = \epsilon_{123} C_{21} = C_{21}. \end{aligned} \quad (282)$$

Moving on to the 'odd' elements corresponding to the first component of the $SU(2)_-$ charge,

$$\begin{aligned} (C_1)^{21} &= C_{be} (\epsilon_{1b2} \delta_{1e} + \epsilon_{eb1} \delta_{12}) = \epsilon_{132} C_{31} = -C_{31}; \\ (C_1)^{32} &= C_{be} (\epsilon_{1b3} \delta_{2e} + \epsilon_{eb2} \delta_{13}) = \epsilon_{123} C_{22} = C_{22}; \\ (C_1)^{13} &= C_{be} (\epsilon_{1b1} \delta_{3e} + \epsilon_{eb3} \delta_{11}) = C_{21} - C_{12} = -C_{[12]}. \end{aligned} \quad (283)$$

Moving on to the 'diagonal' elements corresponding to the first component of the $SU(2)_-$ charge,

³⁷Observe the order of the factors in this product, as well as the convention for lowering the indices.

$$\begin{aligned}
(C_1)^{11} &= C_{be}(\epsilon_{1b1}\delta_{1e} + \epsilon_{eb1}\delta_{11}) = C_{32} - C_{23} = -C_{[23]}; \\
(C_1)^{22} &= C_{be}(\epsilon_{1b2}\delta_{2e} + \epsilon_{eb2}\delta_{12}) = C_{32}\epsilon_{132} = -C_{32}; \\
(C_1)^{33} &= C_{be}(\epsilon_{1b3}\delta_{3e} + \epsilon_{eb3}\delta_{13}) = C_{23}\epsilon_{123} = C_{23}.
\end{aligned} \tag{284}$$

Moving on to the ‘even’ elements corresponding to the second component of the $SU(2)_-$ charge,

$$\begin{aligned}
(C_2)^{12} &= C_{be}(\epsilon_{2b1}\delta_{2e} + \epsilon_{eb2}\delta_{21}) = \epsilon_{231}C_{32} = C_{32}; \\
(C_2)^{23} &= C_{be}(\epsilon_{2b2}\delta_{3e} + \epsilon_{eb3}\delta_{22}) = C_{21} - C_{12} = -C_{[12]}; \\
(C_2)^{31} &= C_{be}(\epsilon_{2b3}\delta_{1e} + \epsilon_{eb1}\delta_{23}) = \epsilon_{213}C_{11} = -C_{11}.
\end{aligned} \tag{285}$$

Moving on to the ‘odd’ elements corresponding to the second component of the $SU(2)_-$ charge,

$$\begin{aligned}
(C_2)^{21} &= C_{be}(\epsilon_{2b2}\delta_{1e} + \epsilon_{eb1}\delta_{22}) = C_{32} - C_{23} = -C_{[23]}; \\
(C_2)^{32} &= C_{be}(\epsilon_{2b3}\delta_{2e} + \epsilon_{eb2}\delta_{23}) = \epsilon_{213}C_{12} = -C_{12}; \\
(C_2)^{13} &= C_{be}(\epsilon_{2b1}\delta_{3e} + \epsilon_{eb3}\delta_{21}) = \epsilon_{231}C_{33} = C_{33}.
\end{aligned} \tag{286}$$

Moving on to the ‘diagonal’ elements corresponding to the second component of the $SU(2)_-$ charge,

$$\begin{aligned}
(C_2)^{11} &= C_{be}(\epsilon_{2b1}\delta_{1e} + \epsilon_{eb1}\delta_{21}) = \epsilon_{231}C_{31} = C_{31}; \\
(C_2)^{22} &= C_{be}(\epsilon_{2b2}\delta_{2e} + \epsilon_{eb2}\delta_{22})C_{13} - C_{31} = -C_{[31]}; \\
(C_2)^{33} &= C_{be}(\epsilon_{2b3}\delta_{3e} + \epsilon_{eb3}\delta_{23}) = C_{13}\epsilon_{213} = -C_{13}.
\end{aligned} \tag{287}$$

And finally, we move on to the elements corresponding to the third component of the $SU(2)_-$ charge. Starting with the ‘even’ components, we have

$$\begin{aligned}
(C_3)^{12} &= C_{be}(\epsilon_{3b1}\delta_{2e} + \epsilon_{eb2}\delta_{31}) = C_{22}\epsilon_{321} = -C_{22}; \\
(C_3)^{23} &= C_{be}(\epsilon_{3b2}\delta_{3e} + \epsilon_{eb3}\delta_{32}) = \epsilon_{312}C_{13} = C_{13}; \\
(C_3)^{31} &= C_{be}(\epsilon_{3b3}\delta_{1e} + \epsilon_{eb1}\delta_{33}) = C_{32} - C_{23} = -C_{[23]}.
\end{aligned} \tag{288}$$

Moving on to the ‘antisymmetric’ elements corresponding to the third component of the $SU(2)_-$ charge,

$$\begin{aligned}
(C_3)^{21} &= C_{be}(\epsilon_{3b2}\delta_{1e} + \epsilon_{eb1}\delta_{32}) = C_{11}\epsilon_{312} = C_{11}; \\
(C_3)^{32} &= C_{be}(\epsilon_{3b3}\delta_{2e} + \epsilon_{eb2}\delta_{33}) = C_{13} - C_{31} = -C_{[31]}; \\
(C_3)^{13} &= C_{be}(\epsilon_{3b1}\delta_{3e} + \epsilon_{eb3}\delta_{31}) = C_{23}\epsilon_{321} = -C_{23}.
\end{aligned} \tag{289}$$

And finally, moving on to the ‘diagonal’ elements corresponding to the third component of the $SU(2)_-$ charge,

$$\begin{aligned}
(C_3)^{11} &= C_{be}(\epsilon_{3b1}\delta_{1e} + \epsilon_{eb1}\delta_{31}) = C_{21}\epsilon_{321} = -C_{21}; \\
(C_3)^{22} &= C_{be}(\epsilon_{3b2}\delta_{2e} + \epsilon_{eb2}\delta_{32})\epsilon_{312}C_{12} = C_{12}; \\
(C_3)^{33} &= C_{be}(\epsilon_{3b3}\delta_{3e} + \epsilon_{eb3}\delta_{33}) = C_{21} - C_{12} = -C_{[12]}.
\end{aligned} \tag{290}$$

9 Appendix B: Results from the standard configurations

9.1 Abelian contribution

We now compute the constituents for the Gauss' law constraint for a wide class of physically interesting configurations. We will display the intermediate steps of the calculation to elucidate on how various structures of interest recur. Take the Ashtekar magnetic field to be of the form³⁸

$$A_i^a = \delta_{ai} \left(\frac{\alpha}{r} \right) + \epsilon_{aij} x^j \left(\frac{\beta}{r^2} \right) + x^a x^i \left(\frac{\gamma}{r^3} \right). \quad (291)$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$ and $\gamma = \gamma(t)$ are spatially constant. First we will compute the Ashtekar magnetic field for (291), given by

$$B_a^i = \epsilon^{ijk} \partial_j A_k^a + \frac{1}{2} \epsilon^{ijk} f^{abc} A_j^b A_k^c. \quad (292)$$

Starting with the terms involving spatial gradients we have

$$\begin{aligned} \epsilon_{ijk} \partial_j A_k^a &= \partial_j \left[\delta_{ak} \left(\frac{\alpha}{r} \right) + \epsilon_{akm} x^m \left(\frac{\beta}{r^2} \right) + x^a x^k \left(\frac{\gamma}{r^3} \right) \right] \\ &= \epsilon_{ijk} \left[\delta_{ak} \left(-\frac{\alpha x^j}{r^3} \right) + \beta \left(\epsilon_{akj} \left(\frac{1}{r^2} \right) - 2\epsilon_{akm} \left(\frac{x^m x^j}{r^4} \right) \right. \right. \\ &\quad \left. \left. + \gamma \left(-\frac{3}{r^5} \right) x^a x^k x^j + \frac{\gamma}{r^3} (\delta_{aj} x^k + \delta_{kj} x^a) \right] \right] \end{aligned} \quad (293)$$

This simplifies to

$$\begin{aligned} &\epsilon^{ija} \left(-\frac{\alpha x^j}{r^3} \right) - 2\delta_{ai} \left(\frac{\beta}{r^2} \right) - \frac{2\beta}{r^4} (\delta_m^i \delta_a^j - \delta_a^i \delta_m^j) x^m x^j \\ &- \left(\frac{3\gamma}{r^5} \right) \epsilon^{ijk} x^a x^k x^j + \frac{\gamma}{r^3} \epsilon^{iak} x^k + \frac{\gamma}{r^3} \epsilon^{ijk} \delta_{kj} x^a. \end{aligned} \quad (294)$$

In the first line of (294) the second term cancels with the second member of the third term in brackets and the fourth and sixth terms, in the second line, vanish due to antisymmetry of ϵ^{ijk} . The result is that the gradient contribution to B_a^i , the ‘Abelian’ part, is given by

$$\epsilon^{iaj} x^j \frac{(\alpha + \gamma)}{r^3} - \left(\frac{2\beta}{r^4} \right) x^i x^a. \quad (295)$$

³⁸We take the connection to have mass dimension of $[A_i^a] = 1$, as expected for a gauge field. This fixes the position dependence of the individual terms.

9.2 Nonabelian contribution

The nonabelian contribution to B_a^i is given by

$$\begin{aligned} \frac{1}{2}\epsilon^{ijk}f^{abc}A_j^bA_k^c &= \frac{1}{2}\epsilon^{ijk}f^{abc}\left[\delta_{bj}\left(\frac{\alpha}{r}\right) + \epsilon_{bjm}x^m\left(\frac{\beta}{r^2}\right) + x^bx^j\left(\frac{\gamma}{r^3}\right)\right] \\ &\times \left[\delta_{ck}\left(\frac{\alpha}{r}\right) + \epsilon_{ckn}x^n\left(\frac{\beta}{r^2}\right) + x^ck^j\left(\frac{\gamma}{r^3}\right)\right]. \end{aligned} \quad (296)$$

Equation (296) splits into nine terms. The first term is given by

$$\frac{1}{2}\epsilon^{ijk}f^{abc}\delta_{ck}\delta_{bj}\left(\frac{\alpha^2}{r^2}\right) = \delta_{ai}\left(\frac{\alpha^2}{r^2}\right). \quad (297)$$

Moving on to the second term of (296) we have

$$\frac{1}{2}\epsilon^{ijk}f^{abc}\delta_{ck}\epsilon_{bjm}x^m\left(\frac{\alpha\beta}{r^3}\right) = \frac{1}{2}(\delta^{ia}\delta^{jb} - \delta^{ib}\delta^{ja})\epsilon_{bjm}x^m\left(\frac{\alpha\beta}{r^3}\right) = -\epsilon_{iam}x^m\left(\frac{\alpha\beta}{2r^3}\right) \quad (298)$$

We have used the vanishing of the first term on the right hand side. Moving on the third term of (296), we have

$$\begin{aligned} \frac{1}{2}\epsilon^{ijk}f^{abc}\delta_{ck}x^bx^j\left(\frac{\alpha\gamma}{r^4}\right) &= \frac{1}{2}(\delta^{ia}\delta^{jb} - \delta^{ib}\delta^{ja})x^bx^j\left(\frac{\alpha\gamma}{r^4}\right) \\ &= \delta^{ai}\left(\frac{\alpha\gamma}{2r^4}\right) - x^ix^a\left(\frac{\alpha\gamma}{2r^4}\right), \end{aligned} \quad (299)$$

where we have used $x^ix^i = r^2$.

The fourth term of (296) is given by

$$\frac{1}{2}\epsilon^{ijk}f^{abc}\epsilon_{ckn}\delta_{bj}x^n\left(\frac{\alpha\beta}{r^3}\right) = \frac{1}{2}\epsilon_{ijk}(\delta_k^a\delta_n^b - \delta_n^a\delta_k^b)\delta_{bj}x^n\left(\frac{\alpha\beta}{r^3}\right) = \epsilon^{ain}x^n\left(\frac{\alpha\beta}{2r^3}\right) \quad (300)$$

Moving on to the fifth term of (296), we have

$$\begin{aligned} &\frac{1}{2}\epsilon^{ijk}f^{abc}\epsilon_{ckn}\epsilon_{bjm}x^n x^m\left(\frac{\delta^2}{r^4}\right) \\ &= \frac{1}{2}(\delta_b^i\delta_m^k - \delta_m^i\delta_b^k)(\delta_k^a\delta_n^b - \delta_n^a\delta_k^b)x^n x^m\left(\frac{\beta^2}{r^4}\right) \\ &= \frac{1}{2}(\delta_b^i\delta_m^k\delta_k^a\delta_n^b - \delta_b^i\delta_m^k\delta_n^a\delta_k^b - \delta_m^i\delta_b^k\delta_k^a\delta_n^b + \delta_m^i\delta_b^k\delta_n^a\delta_k^b)x^ax^m\left(\frac{\beta^2}{r^4}\right) \\ &= \frac{1}{2}(\delta_n^i\delta_m^a - \delta_m^i\delta_n^a - \delta_m^i\delta_n^a + 3\delta_m^i\delta_n^a)x^mx^n\left(\frac{\beta^2}{r^4}\right) = x^ax^i\left(\frac{\beta^2}{r^4}\right). \end{aligned} \quad (301)$$

Moving on to the sixth term of (296), we have

$$\begin{aligned}
& \frac{1}{2} \epsilon^{ijk} f^{abc} \epsilon_{ckn} x^n x^b x^j \left(\frac{\gamma\beta}{r^5} \right) \\
&= \frac{1}{2} \epsilon^{ijk} (\delta_k^a \delta_n^b - \delta_n^a \delta_k^b) x^n x^b x^j \left(\frac{\gamma\beta}{r^5} \right) = \epsilon^{ija} x^j \left(\frac{\gamma\beta}{2r^3} \right). \tag{302}
\end{aligned}$$

The seventh term of (296) is given by

$$\begin{aligned}
\frac{1}{2} \epsilon^{ijk} f^{abc} \delta_{bj} x^c x^k \left(\frac{\alpha\gamma}{r^4} \right) &= \frac{1}{2} (\delta^{ia} \delta^{kc} - \delta^{ic} \delta^{ka}) x^c x^k \left(\frac{\alpha\gamma}{r^4} \right) \\
&= \delta^{ia} \left(\frac{\alpha\gamma}{2r^2} \right) - x^i x^a \left(\frac{\alpha\gamma}{2r^2} \right). \tag{303}
\end{aligned}$$

Moving on to the eight term of (296), we have

$$\begin{aligned}
& \frac{1}{2} \epsilon^{ijk} f^{abc} \epsilon_{bjm} x^m x^c x^k \left(\frac{\beta\gamma}{r^5} \right) \\
&= \frac{1}{2} \epsilon^{ijk} (\delta_m^a \delta_j^c - \delta_j^a \delta_m^c) x^m x^c x^k \left(\frac{\beta\gamma}{r^5} \right) = -\epsilon^{iak} x^k \left(\frac{\beta\gamma}{2r^3} \right) \tag{304}
\end{aligned}$$

where we have use the fact that the first term vanishes. Finally, moving on to the last term of (296) we have

$$\frac{1}{2} \epsilon^{ijk} f^{abc} x^b x^j x^c x^k \left(\frac{\gamma^2}{r^6} \right) = 0. \tag{305}$$

Combining the results of (295) and (297) through (305), we have that the Ashtekar magnetic field for (291) is given by

$$B_a^i = \delta^{ia} \frac{\alpha(\alpha + \gamma)}{r^2} + x^i x^a \left(\frac{-\alpha\gamma + \beta^2}{r^4} \right) - \epsilon^{iam} x^m \frac{\beta(\alpha + \gamma)}{r^3}. \tag{306}$$

The vector fields $\mathbf{v}_a = B^i \partial_i$ for the configuration (291) are given by

$$\mathbf{v}_a = \frac{\alpha(\alpha + \gamma)}{r^2} \frac{\partial}{\partial x^a} + \left(\frac{-\alpha\gamma + \beta^2}{r^4} \right) x^a \frac{d}{dt} - \frac{\beta(\alpha + \gamma)}{r^3} J_a, \tag{307}$$

where $d/dt = x_i \partial_i$ and J_a are the angular momentum operators.

9.3 Magnetic helicity density matrix

The magnetic helicity density matrix C_{ae} , which can be thought of as the direction cosines between the the coordinate axes of A_i^a and B_a^i , is given by

$$\begin{aligned}
C_{ae} = A_i^a B_e^i &= \left[\delta_{ai} \left(\frac{\alpha}{r} \right) + \epsilon_{aij} x^j \left(\frac{\beta}{r^2} \right) + x^a x^i \left(\frac{\gamma}{r^3} \right) \right] \\
&\times \left[\delta_{ie} \frac{\alpha(\alpha + \gamma)}{r^2} + x^i x^e \left(\frac{-\alpha\gamma + \beta^2}{r^4} \right) - \epsilon^{iem} x^m \frac{\beta(\alpha + \gamma)}{r^3} \right]. \quad (308)
\end{aligned}$$

Equation (308) splits into nine terms, given by

$$\begin{aligned}
C_{ae} = & \delta_{ae} \frac{\alpha^2(\alpha + \gamma)}{r^3} + x^a x^e \frac{\alpha(-\alpha\gamma + \beta^2)}{r^5} - \epsilon^{aem} x^m \frac{\alpha\beta(\alpha + \beta)}{r^4} \\
& + \epsilon_{aej} x^j \frac{\alpha\beta(\alpha + \beta)}{r^4} + 0 + (\delta_j^e \delta_a^m - \delta_a^e \delta_j^m) x^j x^m \frac{\beta^2(\alpha + \gamma)}{r^5} \\
& + x^a x^e \frac{\alpha\gamma(\alpha + \gamma)}{r^5} + x^a x^e \frac{\gamma(-\alpha\gamma + \beta^2)}{r^5} - 0. \quad (309)
\end{aligned}$$

Collecting like terms, this reduces to

$$\begin{aligned}
C_{ae} &= \delta^{ae} \frac{(\alpha^2 - \beta^2)(\alpha + \gamma)}{r^3} \\
&+ \frac{x^a x^e}{r^5} \left((\alpha + \beta)(\beta^2 + \alpha\gamma) + (-\alpha\gamma + \beta^2)(\alpha + \gamma) \right) \\
&+ \epsilon^{aem} x^m \left(\frac{\alpha\beta(-\alpha - \beta + \alpha + \gamma)}{r^4} \right) \\
&= \delta^{ae} \frac{(\alpha^2 - \beta^2)(\alpha + \gamma)}{r^3} + 2x^a x^e \frac{\beta^2(\alpha + \gamma)}{r^5} + \epsilon^{aem} x^m \frac{\alpha\beta(\gamma - \alpha)}{r^4}. \quad (310)
\end{aligned}$$

10 Appendix C: Roots of the cubic polynomial in closed form

We would like to solve the cubic equation

$$\epsilon^3 + A\epsilon^2 + B\epsilon + C = 0 \quad (311)$$

for some coefficients A , B and C . First we perform the transformation $\epsilon = y + r$, yielding

$$\begin{aligned} & (y + r)^3 + A(y + r)^2 + B(y + r) + C \\ &= y^3 + (3r + A)y^2 + (3r^2 + 2Ar + B)y + r^3 + Ar^2 + Br + C = 0. \end{aligned} \quad (312)$$

By the method of Cardano we choose $r = -\frac{A}{3}$ in order to eliminate the quadratic term in (312), which yields

$$y^3 + py = q \quad (313)$$

where

$$p = B - \frac{A^2}{3}; \quad q = -\left(C - \frac{AB}{3} + \frac{2A^3}{27}\right) \quad (314)$$

making the further substitution $y = (4p/3)^{1/3}m$, we have

$$m^3 + \frac{3}{4}m = q\left(\frac{4p}{3}\right)^{-3/2} \quad (315)$$

Letting $m = \sin\theta$ and making use of the identity $4\sin^3\theta + 3\sin\theta = \sin 3\theta$, we have

$$\sin 3\theta = q\left(\frac{4p}{3}\right)^{-3/2}; \quad \theta = \frac{1}{3}\sin^{-1}\left(4q(4p/3)^{-3/2} + \rho\right) \quad (316)$$

where $\rho = \frac{n\pi}{3}$ labels the three roots for $n = (0, \pm 1)$. So the solution of (315) is given by

$$m = \sin\theta = T_{1/3}^\rho\left[4q(4p/3)^{-3/2}\right], \quad (317)$$

where we have defined $T_\alpha^\rho(x) = \sin[\alpha\sin^{-1}x + \rho]$. This leads to the general solution to (311) of

$$\epsilon = -\frac{A}{3} + \sqrt{\frac{4}{3}\left(B - \frac{A^2}{3}\right)} T_{1/3}^\rho\left[\frac{-4\left(C - \frac{AB}{3} + \frac{2A^3}{27}\right)}{\left(\frac{4}{3}\left(B - \frac{A^2}{3}\right)\right)^{3/2}}\right] \quad (318)$$

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